International Journal of Bridge Engineering (IJBE), Vol. 2, No. 2, (2014), pp. 31-47

STABILITY OF CURVED-IN-PLANE C-S BRIDGES

Ioannis G. Raftoyiannis[1], George T. Michaltsos[1]

[1] National Technical University of Athens, Dept. of Civil Engineering, Athens, Greece
e-mail: rafto@central.ntua.gr, michalts@central.ntua.gr

ABSTRACT: In this work, the stability of curved-in-plane cable-stayed bridges is thoroughly studied. Expressing the tensile forces of the cables with respect to the deck and pylon deformations, the cable-stayed bridge problem is reduced to the solution of a curved-in-plane beam representing the deck. A three-dimensional formulation is considered for the analysis of the c-s bridge model. The theoretical formulation is based on a continuum approach, which has been widely employed in the literature to analyze long span bridges. Two case studies are carried out in the present work. The first is concerned with the determination of the critical sectorial angle of an unloaded bridge and the second one with the determination of the critical horizontal load related to the sectorial angle of the bridge.

KEYWORDS: Cable-stayed bridges, Curved beams, Stability, Sectorial angle

1 INTRODUCTION

Cable-stayed bridges are a particular case of bridge structures that have been of great interest in recent years (though they have been known since the beginning of the 18th century), particularly because of their special shape, aesthetic and also, because they are an alternative solution to suspension bridges for long spans (Troitsky [1]).

The study of the behavior of cable-stayed bridges and their substructures reveals several nonlinear behavior patterns, concurrently under normal design loads, due to the individual nonlinearity of the above substructures such as pylons, stay-cables and bridge-deck and their interactions.

There are the geometric nonlinearities that arise, mainly, from the large displacements of cables and the structural nonlinearities that are caused by the strong axial and lateral forces acting on the bridge-deck and the pylons.

There are numerous publications dealing with the static and mainly the dynamic behavior of cable-stayed bridges that have presented significant results (see the main bibliography in Ref[2]).

A special form of cable-stayed bridges is the curved-in-plane ones. In this particular field, the bibliography is rather poor.

The most interesting publications are those of J. Brownjohn et al [3] who study

the dynamic behavior of a 100m span curved cable stayed bridge constructed in Singapore based on full-scale testing and analytical models, and of Sirigorino & Fujino [4] who try to access the dynamic characteristics of the 455m Katsushika harp-type curved cable-stayed bridge by employing a time-domain multi-input multi-output (MIMO) system identification (SI) technique.
The study of the stability of C-S-Bridges curved-in-plane is terra incognita.
In this paper, using a previous publication of authors, we try to study the stability of such a bridge.
Expressing the tensile forces of the cables in relation to the deck and pylon deformations, the problem is reduced to the solution of a beam curved-in-plane. A three-dimensional analysis is considered for the solution of the bridge model. The theoretical formulation is based on a continuum approach, which has been used in the literature to analyze long span bridges. Two cases are carried out. The first is the determination of the critical central angle of an unloaded bridge and the second is the determination of the critical horizontal load in relation to the central angle of the bridge.

2 BASIC RELATIONS

Let us consider the cable-stayed bridge shown in Fig. 1, (a) in a perspective sketch, (b) in plan view, and (c) in front view. The bridge, the deck of which is curved-in-plane with radius of curvature R, is suspended and supported by μ-cables starting from point 1 with an angle ρ_1, and ending at point 2 with an angle ρ_2, and anchored at the top of the pylon PG.
There is also a back-stay cable (single cable or system of cables), as it is shown in Figs 1(a) and 1(c). The deck is made from homogeneous and isotropic material with modulus of elasticity E, and it is a part of a circle with center K, radius R, and sectorial angle ρ.
Thus, its length L is determined by the above angle ρ through the relation:

$$L = R \cdot \rho \tag{1a}$$

The pylon is made from homogeneous and isotropic material with modulus of elasticity E_p, while it is inclined as to the vertical (PP') at angle γ_1 and has a length (PG) given by the relation:

$$PG = \frac{h}{\cos\gamma_1} \tag{1b}$$

where h is the distance of the top of the pylon P from the foundation level.
We consider in addition that the cross-section of the pylon is referred to the main-axes 1-1 and 2-2, and that the pylon and the back-stay cable are located on the vertical plane that contains the main-axis 2-2. The cables are made from material with modulus of elasticity E_c.
The back-stay cable is inclined with respect to the vertical direction by an angle γ_2 and its length is given by the relation:

$$PC = \frac{h}{\cos\gamma_2} \tag{1c}$$

where it is assumed that both the pylon and the back-stay cable are founded on the same level without loss of generality.
The plane of the deck is at a distance h_o above the foundation level.
The projection P' of the top of the pylon on the level of the deck is determined by the lengths ℓ_1 and ℓ_2 (Fig. 1) which are known. Easily, one can determine the angles β_1 and β_2 through the relations:

$$\cos\beta_1 = \frac{R^2 + \ell_1^2 - \ell_2^2}{2\ell_1 R} \quad , \quad \cos\beta_2 = \frac{R^2 + \ell_2^2 - \ell_1^2}{2\ell_2 R} \tag{1d}$$

The deck-beam is referred to the three-orthogonal, clockwise, curvilinear coordinates system A,x,y,z shown in Fig. 1(a).
With the assumptions and the analysis presented by Raftoyiannis & Michaltsos [2], the equations governing the behavior of the bridge under static loads are the following:

$$EAu'' + \frac{EA}{R^2}u = -p_x(x) + F(x)\sin\xi\sin\delta$$

$$EJ_z\upsilon'''' + \frac{2EJ_z}{R^2}\cdot\upsilon'' + \frac{EJ_z}{R^4}\cdot\upsilon = p_y(x) - F(x)\cos\xi\sin\delta$$

$$\left(EJ_y - \frac{EJ_\omega}{R^2}\right)w'''' + \frac{GJ_d}{R^2}\cdot w'' - \frac{EJ_y - GJ_d}{R}\cdot\varphi'' - \frac{EJ_\omega}{R}\cdot\varphi'''' = p_z(x) - F(x)\cos\delta$$

$$EJ_\omega\varphi'''' - GJ_d\varphi'' - \frac{EJ_y}{R^2}\cdot\varphi + \frac{EJ_\omega}{R}\cdot w'''' + \frac{EJ_y - GJ_d}{R}\cdot w'' = m_x(x) - F(x)(e_y\cos\delta + e_z\cos\xi\sin\delta)$$

(2.a)

where:

$$S_1 = EI_y - \frac{EI_\omega}{R^2}\ , \quad S_2 = \frac{EI_y - GI_d}{R} \tag{2.b}$$

and

$$\begin{aligned} F(x) = {} & A_1(x)u(x) + A_2(x)\upsilon(x) + A_3W(x) \\ & + c(x)\int_{x_1}^{x_2}[\Gamma_1(x)u(x) + \Gamma_2(x)\upsilon(x) + \Gamma_3(x)w(x)]dx \\ & + \varepsilon(x)\int_{x_1}^{x_2}[E_1(x)u(x) + E_2(x)\upsilon(x) + E_3(x)w(x)]dx \end{aligned} \tag{2.c}$$

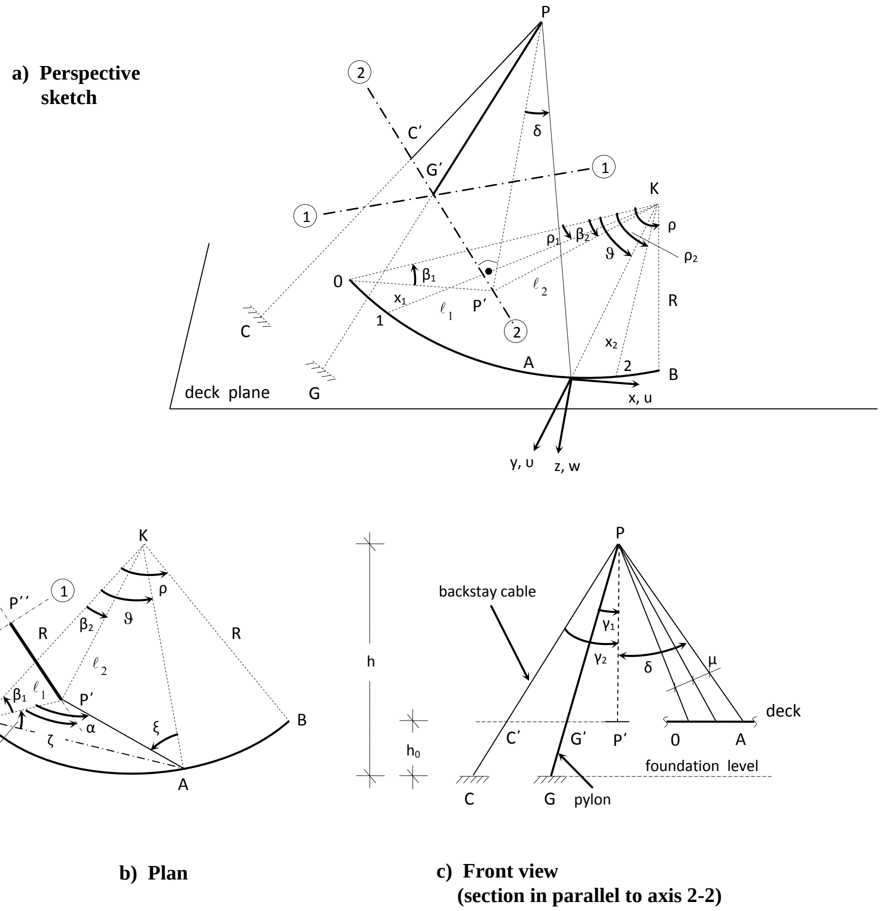

Fig. 1 Perspective sketch, plan, and front view of a curved-in-plane c-s bridge

In addition we have [2]:

$$A_1(x) = \frac{E_c A(x)}{s(x)} \cdot \sin\xi \sin\delta$$

$$A_2(x) = \frac{E_c A(x)}{s(x)} \cdot \cos\xi \sin\delta$$

$$A_3(x) = \frac{E_c A(x)}{s(x)} \cdot \cos\delta$$

$$\Gamma_1(x) = \overline{B}_1(x)d(x)+\overline{B}_2(x)r(x)\ \sin\xi\sin\delta$$
$$\Gamma_2(x) = \overline{B}_1(x)d(x)+\overline{B}_2(x)r(x)\ \cos\xi\sin\delta \qquad (2.d)$$
$$\Gamma_3(x) = \overline{B}_1(x)d(x)+\overline{B}_2(x)r(x)\ \cos\delta$$
$$E_1(x) = \overline{B}_3(x)d(x)+\overline{B}_4(x)r(x)\ \sin\xi\sin\delta$$
$$E_2(x) = \overline{B}_3(x)d(x)+\overline{B}_4(x)r(x)\ \cos\xi\sin\delta$$
$$E_3(x) = \overline{B}_3(x)d(x)+\overline{B}_4(x)r(x)\ \cos\delta$$

$$\overline{B}_1 = \frac{1}{\overline{B}_5}\cdot\left(1+\int_{x_1}^{x_2}\varepsilon(x)r(x)dx\right) \quad , \quad \overline{B}_2 = -\frac{1}{\overline{B}_5}\cdot\int_{x_1}^{x_2}\varepsilon(x)d(x)dx$$

$$\overline{B}_3 = -\frac{1}{\overline{B}_5}\cdot\int_{x_1}^{x_2}c(x)r(x)dx \quad , \quad \overline{B}_4 = \frac{1}{\overline{B}_5}\cdot\left(1+\int_{x_1}^{x_2}c(x)d(x)dx\right)$$

$$\overline{B}_5 = \left(1+\int_{x_1}^{x_2}c(x)d(x)dx\right)\cdot\left(1+\int_{x_1}^{x_2}\varepsilon(x)r(x)dx\right)-\int_{x_1}^{x_2}c(x)r(x)dx\cdot\int_{x_1}^{x_2}\varepsilon(x)d(x)dx$$

(2.e)

$$c(x) = \frac{E_c A(x)}{s(x)}\cdot\frac{h^3}{3E_p J_2}\cdot\frac{\sin(\alpha-\zeta)\sin\delta}{\cos^3\gamma_1}$$
$$d(x) = \sin(\alpha-\zeta)\sin\delta$$
$$\varepsilon(x) = \frac{E_c A(x)}{s(x)}\cdot\frac{h^3\cdot K}{3E_p J_1}\cdot\frac{\cos(\alpha-\zeta)\sin\delta}{\cos^3\gamma_1}$$
$$r(x) = \cos(\alpha-\zeta)\sin\delta$$

$$\cos\alpha = \frac{\ell_1 - 2R\sin\frac{\vartheta}{2}\sin(\frac{\vartheta}{2}+\beta_1)}{\sqrt{\ell_1^2+4R^2\sin^2\frac{\vartheta}{2}-4\ell_1 R\sin\frac{\vartheta}{2}\sin(\frac{\vartheta}{2}+\beta_1)}}$$
$$\tan\delta = \frac{\sqrt{\ell_1^2+4R^2\sin^2\frac{\vartheta}{2}-4\ell_1 R\sin\frac{\vartheta}{2}\sin(\frac{\vartheta}{2}+\beta_1)}}{h-h_o}$$
$$\xi = \alpha-\beta_1-\vartheta \qquad (2.f)$$
$$s_i = \frac{h-h_o}{\cos\delta}$$
$$\vartheta = \frac{x}{R}$$
$$K = 1+\frac{h^3 E_c A_{bc}\cdot\sin^2(\gamma_2-\gamma_1)}{3E_p J_1\cdot s_{bc}\cdot\cos^4\gamma_1}$$

In Fig. 2, one can see the deformed state of the bridge as well as the deformations of the deck.

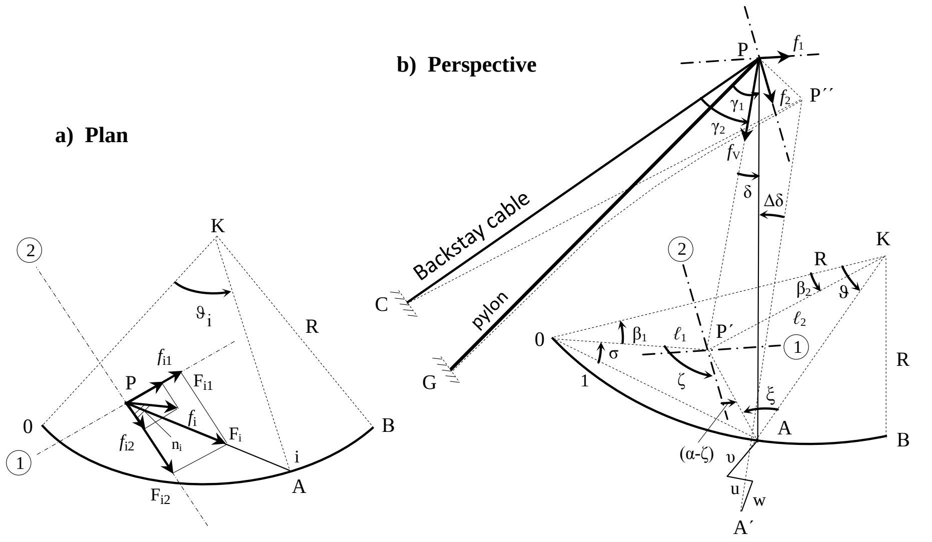

Fig. 2 The deformed state of the bridge

3 THE UNLOADED BRIDGE

For the unloaded bridge we have:

$$p_x = p_y = p_z = m_x = 0 \tag{3.a}$$

In order to solve the system of equations (2.a), we are searching for a solution of the form:

$$\begin{aligned} u(x) &= \sum_n U_n \sin\frac{n\pi x}{L} \\ \upsilon(x) &= \sum_n V_n \sin\frac{n\pi x}{L} \\ w(x) &= \sum_n W_n \sin\frac{n\pi x}{L} \\ \varphi(x) &= \sum_n \Phi_n \sin\frac{n\pi x}{L} \end{aligned} \tag{3.b}$$

Introducing the expressions of eqs (3.b) into eqs (2.a), we get:

$$-EA\sum_n U_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}+\frac{EA}{R^2}\sum_n U_n\sin\frac{n\pi x}{L}=\bar{F}(x)\sin\xi\sin\delta$$

$$EI_z\sum_n V_n\left(\frac{n\pi}{L}\right)^4\sin\frac{n\pi x}{L}-\frac{2EI_z}{R^2}\sum_n V_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}+\frac{EI_z}{R^4}\sum_n V_n\sin\frac{n\pi x}{L}=-\bar{F}(x)\cos\xi\sin\delta$$

$$S_1\sum_n W_n\left(\frac{n\pi}{L}\right)^4\sin\frac{n\pi x}{L}-\frac{GI_d}{R^2}\sum_n W_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}+S_2\sum_n\Phi_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}$$
$$-\frac{EI_\omega}{R^2}\sum_n\Phi_n\left(\frac{n\pi}{L}\right)^4\sin\frac{n\pi x}{L}=-\bar{F}(x)\cos\delta$$

$$EI_\omega\sum_n\Phi_n\left(\frac{n\pi}{L}\right)^4\sin\frac{n\pi x}{L}+GI_d\sum_n\Phi_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}-\frac{EI_y}{R^2}\sum_n\Phi_n\sin\frac{n\pi x}{L}$$
$$+\frac{EI_\omega}{R}\sum_n W_n\left(\frac{n\pi}{L}\right)^4\sin\frac{n\pi x}{L}-S_2\sum_n W_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}=-\bar{F}(x)(e_y\cos\delta+e_z\cos\xi\sin\delta)$$

(4.a)

where:

$$\bar{F}(x)=A_1(x)\sum_n U_n\sin\frac{n\pi x}{L}+A_2(x)\sum_n V_n\sin\frac{n\pi x}{L}+A_3(x)\sum_n W_n\sin\frac{n\pi x}{L}$$
$$+c(x)\int_{x_1}^{x_2}\left(\Gamma_1(x)\sum_n U_n\sin\frac{n\pi x}{L}+\Gamma_2(x)\sum_n V_n\sin\frac{n\pi x}{L}+\Gamma_3(x)\sum_n W_n\sin\frac{n\pi x}{L}\right)dx$$
$$+\varepsilon(x)\int_{x_1}^{x_2}\left(E_1(x)\sum_n U_n\sin\frac{n\pi x}{L}+E_2(x)\sum_n V_n\sin\frac{n\pi x}{L}+E_3(x)\sum_n W_n\sin\frac{n\pi x}{L}\right)dx$$

(4.b)

Multiplying successively each of eqs(4.a) by $\sin\frac{k\pi x}{L}$ (k=1 to n) and taking into account the orthogonality condition, we conclude to the following system:

$$\begin{aligned}
&a_{uk1}U_1+\cdots+a_{ukn}U_n+b_{uk1}V_1+\cdots+b_{ukn}V_n+\gamma_{uk1}W_1+\cdots+\gamma_{ukn}W_n+\delta_{uk1}\Phi_1+\cdots+\delta_{ukn}\Phi_n=0\\
&a_{vk1}U_1+\cdots+a_{vkn}U_n+b_{vk1}V_1+\cdots+b_{vkn}V_n+\gamma_{vk1}W_1+\cdots+\gamma_{vkn}W_n+\delta_{vk1}\Phi_1+\cdots+\delta_{vkn}\Phi_n=0\\
&a_{wk1}U_1+\cdots+a_{wkn}U_n+b_{wk1}V_1+\cdots+b_{wkn}V_n+\gamma_{wk1}W_1+\cdots+\gamma_{wkn}W_n+\delta_{wk1}\Phi_1+\cdots+\delta_{wkn}\Phi_n=0\\
&a_{\varphi k1}U_1+\cdots+a_{\varphi kn}U_n+b_{\varphi k1}V_1+\cdots+b_{\varphi kn}V_n+\gamma_{\varphi k1}W_1+\cdots+\gamma_{\varphi kn}W_n+\delta_{\varphi k1}\Phi_1+\cdots+\delta_{\varphi kn}\Phi_n=0
\end{aligned}$$

(5.a)

where we have considered symmetric hanging of the deck on the cross-section's axis z.

Thus, it will be (for e_y=0):

$$
\begin{aligned}
a_{uki} &= \int_0^L A_1(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx \\
&\quad + \int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}\Gamma_1(x)\cdot\sin\frac{i\pi x}{L}dx \\
&\quad + \int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}E_1(x)\cdot\sin\frac{i\pi x}{L}dx \\
a_{ukk} &= \left[EA\left(\frac{k\pi}{L}\right)^2-\frac{EA}{R^2}\right]\cdot\frac{L}{2}+\int_0^L A_1(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx \\
&\quad + \int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}\Gamma_1(x)\cdot\sin\frac{i\pi x}{L}dx \\
&\quad + \int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}E_1(x)\cdot\sin\frac{i\pi x}{L}dx \\
b_{uki} &= \int_0^L A_2(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx \\
&\quad + \int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}\Gamma_2(x)\cdot\sin\frac{i\pi x}{L}dx \\
&\quad + \int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}E_2(x)\cdot\sin\frac{i\pi x}{L}dx \\
\gamma_{uki} &= \int_0^L A_3(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx \\
&\quad + \int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}\Gamma_3(x)\cdot\sin\frac{i\pi x}{L}dx \\
&\quad + \int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\sin\xi\cdot\sin\delta\, dx\cdot\int_{x_1}^{x_2}E_3(x)\cdot\sin\frac{i\pi x}{L}dx \\
\delta_{uki} &= 0
\end{aligned}
\tag{5.b}
$$

$$
\begin{aligned}
a_{vki} &= \int_0^L A_1(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\\
&\quad+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}\Gamma_1(x)\cdot\sin\frac{i\pi x}{L}dx\\
&\quad+\int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}E_1(x)\cdot\sin\frac{i\pi x}{L}dx\\
b_{vki} &= \int_0^L A_2(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\\
&\quad+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}\Gamma_2(x)\cdot\sin\frac{i\pi x}{L}dx\\
&\quad+\int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}E_2(x)\cdot\sin\frac{i\pi x}{L}dx\\
b_{vkk} &= \left[EI_z\left(\frac{k\pi}{L}\right)^4-\frac{2EI_z}{R^2}\left(\frac{k\pi}{L}\right)^2+\frac{EI_z}{R^4}\right]\cdot\frac{L}{2}+\int_0^L A_2(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\\
&\quad+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}\Gamma_2(x)\cdot\sin\frac{i\pi x}{L}dx\\
&\quad+\int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}E_2(x)\cdot\sin\frac{i\pi x}{L}dx\\
\gamma_{vki} &= \int_0^L A_3(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\\
&\quad+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}\Gamma_3(x)\cdot\sin\frac{i\pi x}{L}dx\\
&\quad+\int_0^L \varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\xi\cdot\sin\delta\,dx\cdot\int_{x_1}^{x_2}E_3(x)\cdot\sin\frac{i\pi x}{L}dx\\
\delta_{vki} &= 0
\end{aligned}
\tag{5.c}
$$

$$a_{wki}=\int_0^L A_1(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_1(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}E_1(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$b_{wki}=\int_0^L A_2(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_2(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}E_2(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$\gamma_{wki}=\int_0^L A_3(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_3(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}E_3(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$\gamma_{wkk}=\left[S_1\left(\frac{k\pi}{L}\right)^4-\frac{GI_d}{R^2}\left(\frac{k\pi}{L}\right)^2\right]\cdot\frac{L}{2}+\int_0^L A_3(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_3(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot\cos\delta dx\cdot\int_{x_1}^{x_2}E_3(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$\delta_{wki}=0$$
$$\delta_{wkk}=\left[S_2\left(\frac{k\pi}{L}\right)^2-\frac{EI_\omega}{R^2}\left(\frac{k\pi}{L}\right)^4\right]\cdot\frac{L}{2}\tag{5.d}$$
$$a_{\varphi ki}=\int_0^L A_1(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_1(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}E_1(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$b_{\varphi ki}=\int_0^L A_2(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_2(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}E_2(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$\gamma_{\varphi ki}=\int_0^L A_3(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}\Gamma_3(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}E_3(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$\gamma_{\varphi ki}=\left[\frac{EI_\omega}{R}\left(\frac{k\pi}{L}\right)^4-S_2\left(\frac{k\pi}{L}\right)^2\right]\cdot\frac{L}{2}+\int_0^L A_3(x)\cdot\sin\frac{i\pi x}{L}\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx$$
$$+\int_0^L c(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y os\delta dx\cdot\int_{x_1}^{x_2}\Gamma_3(x)\cdot\sin\frac{i\pi x}{L}dx+\int_0^L\varepsilon(x)\cdot\sin\frac{k\pi x}{L}\cdot e_y\cos\delta dx\cdot\int_{x_1}^{x_2}E_3(x)\cdot\sin\frac{i\pi x}{L}dx$$
$$\delta_{\varphi ki}=0$$
$$\delta_{\varphi kk}=\left[\frac{EI_\omega}{R}\left(\frac{k\pi}{L}\right)^4-GI_d\left(\frac{k\pi}{L}\right)^2-\frac{EI_y}{R^2}\right]\cdot\frac{L}{2}\tag{5.e}$$

In order for the above in (5.a) linear homogeneous without second member system to have non-trivial solutions, the determinant of the unknowns U_n, V_n, W_n, Φ_n must be equal to zero.

The above-mentioned condition concludes to the following equation:

$$\left\| \Delta_{ik} \right\| = 0 \tag{6}$$

Equation (6) gives the spectrum of the critical values for the buckling angle ρ of the unloaded bridge.

4 THE CASE OF AN EXTERNAL LOAD

Let us consider now that the bridge is loaded by the horizontal load q_y, applied on the bridge as it is shown in figure 3.
The reactions at the supports are:

$$P = q_y \cdot R \tag{7}$$

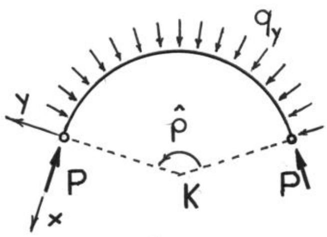

Fig. 3 The acting external load

Therefore, the external loadings of equations (2.a), can be determined as follows:

$$p_y(x) = -P \cdot \upsilon'' - P \cdot z_M \cdot \varphi''$$

$$p_z(x) = -P \cdot w'' + P \cdot y_M \cdot \varphi''$$

$$m_x(x) = -P \cdot z_M \cdot \upsilon'' + P \cdot y_M \cdot w''$$

The above for a cross-section symmetric about axis z (i.e. y_M=0), become:

$$\begin{aligned} p_y(x) &= -P \cdot \upsilon'' - P \cdot z_M \cdot \varphi'' \\ p_z(x) &= -P \cdot w'' \\ m_x(x) &= -P \cdot z_M \cdot \upsilon'' \end{aligned} \tag{8.a}$$

Therefore, equations (2.a), because the above and eq.(7), become:

$$\begin{aligned} & EAu'' + \frac{EA}{R^2} u = F(x) \sin\xi \sin\delta \\ & EJ_z \upsilon'''' + \frac{2EJ_z}{R^2} \cdot \upsilon'' + \frac{EJ_z}{R^4} \cdot \upsilon + q_y z_M R \varphi'' = -F(x) \cos\xi \sin\delta \\ & S_1 \cdot w'''' + \left(\frac{GJ_d}{R^2} + q_y R \right) \cdot w'' - (S_2 - q_y z_M R) \cdot \varphi'' - \frac{EJ_\omega}{R} \cdot \varphi'''' = -F(x) \cos\delta \\ & EJ_\omega \varphi'''' - GJ_d \varphi'' - \frac{EJ_y}{R^2} \cdot \varphi + \frac{EJ_\omega}{R} \cdot w'''' + S_2 \cdot w'' + q_y z_M R \upsilon'' = -F(x) \cdot e_z \cos\xi \sin\delta \end{aligned} \tag{8.b}$$

We search again for a solution under the form of eqs (3.b), and following a procedure like the one outlined in §3, we conclude to the system of eqs (5.a) with coefficients given by relation (5.b) to (5.f), with the following differences:

$$b_{vkk} = \left[EI_z \left(\frac{k\pi}{L} \right)^4 - \left(\frac{2EI_z}{R^2} + q_y R \right) \left(\frac{k\pi}{L} \right)^2 + \frac{EI_z}{R^4} \right] \cdot \frac{L}{2} + \int_0^L A_2(x) \cdot \sin\frac{i\pi x}{L} \cdot \sin\frac{k\pi x}{L} \cdot \cos\xi \cdot \sin\delta\, dx$$

$$+ \int_0^L c(x) \cdot \sin\frac{k\pi x}{L} \cdot \cos\xi \cdot \sin\delta\, dx \cdot \int_{x_1}^{x_2} \Gamma_2(x) \cdot \sin\frac{i\pi x}{L} dx$$

$$+ \int_0^L \varepsilon(x) \cdot \sin\frac{k\pi x}{L} \cdot \cos\xi \cdot \sin\delta\, dx \cdot \int_{x_1}^{x_2} E_2(x) \cdot \sin\frac{i\pi x}{L} dx$$

$$\delta_{vki} = q_y z_M R \left(\frac{k\pi}{L} \right)^2 \frac{L}{2} \tag{9.a}$$

$$\gamma_{wkk} = \left[S_1 \left(\frac{k\pi}{L} \right)^4 - \left(\frac{GI_d}{R^2} + q_y R \right) \left(\frac{k\pi}{L} \right)^2 \right] \cdot \frac{L}{2} + \int_0^L A_3(x) \cdot \sin\frac{i\pi x}{L} \cdot \sin\frac{k\pi x}{L} \cdot \cos\delta\, dx$$

$$+ \int_0^L c(x) \cdot \sin\frac{k\pi x}{L} \cdot \cos\delta\, dx \cdot \int_{x_1}^{x_2} \Gamma_3(x) \cdot \sin\frac{i\pi x}{L} dx \tag{9.b}$$

$$+ \int_0^L \varepsilon(x) \cdot \sin\frac{k\pi x}{L} \cdot \cos\delta\, dx \cdot \int_{x_1}^{x_2} E_3(x) \cdot \sin\frac{i\pi x}{L} dx$$

$$\delta_{wkk} = \left[S_2 - q_y z_M R \left(\frac{k\pi}{L} \right)^2 - \frac{EI_\omega}{R^2} \left(\frac{k\pi}{L} \right)^4 \right] \cdot \frac{L}{2}$$

$$b_{\varphi ki} = -q_y z_M \left(\frac{k\pi}{L} \right)^2 \frac{L}{2} + \int_0^L A_2(x) \cdot \sin\frac{i\pi x}{L} \cdot \sin\frac{k\pi x}{L} \cdot e_y \cos\delta\, dx$$

$$+ \int_0^L c(x) \cdot \sin\frac{k\pi x}{L} \cdot e_y \cos\delta\, dx \cdot \int_{x_1}^{x_2} \Gamma_2(x) \cdot \sin\frac{i\pi x}{L} dx \tag{9.c}$$

$$+ \int_0^L \varepsilon(x) \cdot \sin\frac{k\pi x}{L} \cdot e_y \cos\delta\, dx \cdot \int_{x_1}^{x_2} E_2(x) \cdot \sin\frac{i\pi x}{L} dx$$

In order for the above linear homogeneous system in eq. (5.a) to have non-trivial solution, the determinant of the unknowns U_n, V_n, W_n, Φ_n must be equal to zero.
This condition concludes to the following equation:

$$\left\| \Delta_{ik} \right\| = 0 \tag{10}$$

Equation (10), for different values of n, gives the spectrum of the critical buckling loads q_y.

5 NUMERICAL RESULTS AND DISCUSSION

In this section, a number of numerical case-studies based on the equations obtained in the previous paragraphs have been examined.

Let us consider four bridges with radii of curvature: R=50m, 100m, 200m, and 300m.

Each of the above bridges has stayed-cables whose first edge is anchored onto the deck, starting from the point at x_1=L/10 and ending at the point x_2=9L/10, while the other edge is anchored at the top of a pylon whose position is determined by the length $\ell_1 = 100\,\mathrm{m}$, the angle $\beta_1 = \pi/3$, the heights h=150 m, h_o=50 m, and its inclination by angle $\gamma_1 = \pi/6$. A back-stay cable is applied at the top of the pylon at angle $\gamma_2 = \pi/4$, while its other edge is anchored on the same level with the pylon's foundation. The pylon is designed to be located eccentrically, having the projection of the anchorage point of the cables near to the first quarter (L/4) of the bridge, in order for us to study the influence of this eccentricity on the deformations of the deck.

For the present analysis, concerning the law of the cables cross-sections change, we will adopt the one proposed by Bruno and Golotti [5], analogously modified for the present case of a curved in plane c-s bridge: $A(x) = \dfrac{g}{\sigma_g \cdot \cos\delta}$, where: g is the uniformly distributed deck's own load, σ_g is the initial tension of the stays' curtain due to the above g. It is $\sigma_g = \sigma_\alpha \cdot \dfrac{g}{g+p}$, where σ_a is the allowable stress of the cables (in this example σ_a =12,000 dN/cm^2) and p is the design live load (in this example p=g)

We consider, in addition, a set of two decks' cross-sections, a slender (C-S 1), and a stiff one (C-S 2) the data of which are given in Table 1, and made from steel S460M (with $\sigma_f = 460\,\mathrm{N/cm^2}$). We consider finally that we have a central anchorage with e_z= e_y =0 .

Table 1. Decks' and pylons' properties

	m	A	J_y	J_z	J_d	J_{px}	J_ω	A_p	J_1	J_2	A_{bc}
C-S 1	157	0.20	0.4	6	$1.2*10^{-5}$	3	6	0.20	4	2	0.005
C-S 2	550	0.70	1	10	$1.2*10^{-4}$	6	10	0.20	4	2	0.015

5.1 The unloaded bridge

Applying the expressions of §3, and for any combination of the above data, we find that the critical angle is always $\rho_{critical} \cong 3.14\,\mathrm{rad}$. In figure 4, we see the plot of the determinant of equation 6 for the case C-S1 and $\rho = 3.13 \text{ to } 3.16 \text{ rad}$.

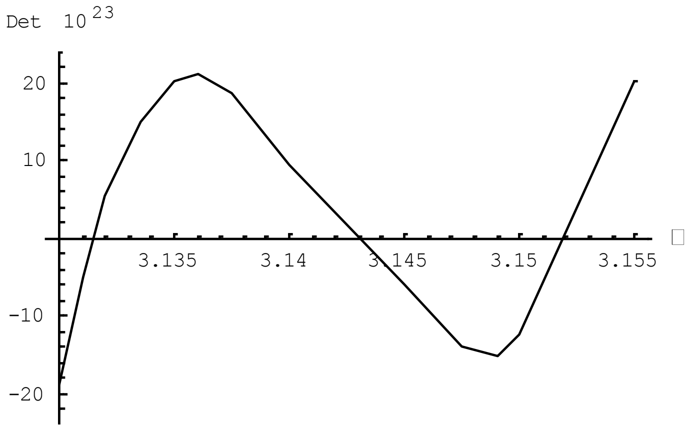

Fig. 4 Determination of the critical angle ρ

5.2 The case of an external load

Let us consider now that the bridge is loaded by the load q_y , as it is shown in figure 3. Applying the formulae of §4, we obtain the following plots, where it is also shown, by a straight line, the load which plasticize the cross-section of the bridge.

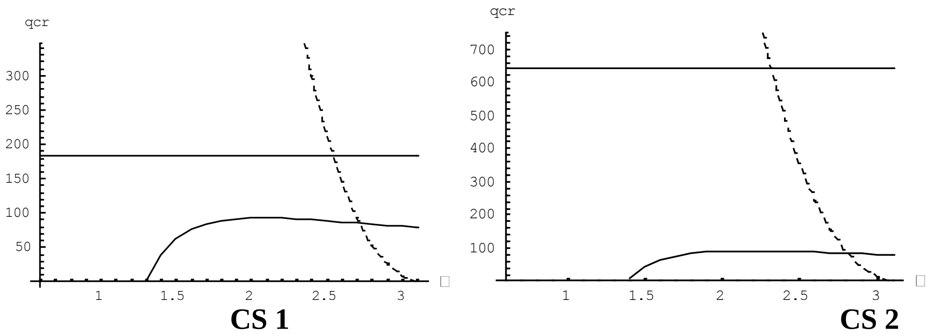

Fig. 5 Critical loads q_y in relation to the angle ρ, for R=50m

Figure 5 shows that for R=50m and the slender cross-section CS 1, the critical external load q_y increases up to $\rho \approx 2.7 \text{ rad}$, while for $\rho > 2.7 \text{ rad}$ the critical load decreases.

For the stiff cross-section CS 2, we see that the critical external load q_y

increases up to $\rho \approx 2.82\,\text{rad}$, while for $\rho > 2.82$ rad the critical load decreases.

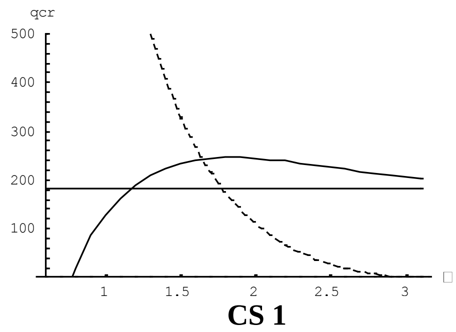

Fig. 6 Critical loads q_y in relation to the angle ρ, for R=100m

Figure 6 shows that for R=100m and the slender cross-section CS 1, the critical external load q_y increases up to $\rho \approx 1.2$ rad. For $\rho \approx 1.2$ rad up to $\rho \approx 1.75$ rad, the critical load is the one that plasticizes the bridge's cross-section, while for $\rho > 1.75$ rad the critical external load q_y decreases.

For the stiff cross-section CS 2, the critical external load q_y increases up to $\rho \approx 0.92$ rad. For $\rho \approx 0.92$ rad up to $\rho \approx 2.27$ rad, the critical load is the one which plasticizes the bridge's cross-section, while for $\rho > 2.27$ rad the critical external load q_y decreases.

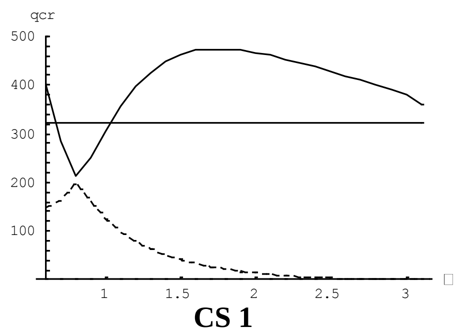

Fig. 7 Critical loads q_y in relation to the angle ρ, for R=200m

Figure 7 shows that for R=200m and the slender cross-section CS 1, the critical external load q_y increases up to $\rho \approx 0.80\,\text{rad}$, while for $\rho > 0.80$ rad the critical load decreases.

For the stiff cross-section CS 2, up to $\rho \approx 1.70$ rad, the critical load is the one which plasticizes the bridge's cross-section, while for $\rho > 1.70$ rad the critical external load q_y decreases.

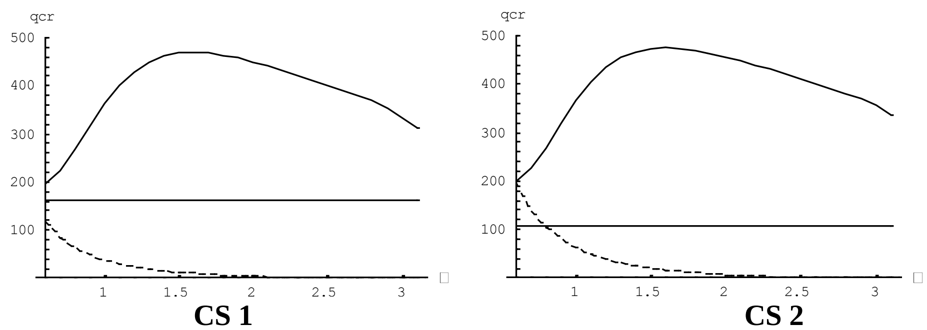

Fig. 8 Critical loads q_y in relation to the angle ρ, for R=300m

Figure 8 shows that for R=300m and the slender cross-section CS 1, the critical external load q_y decreases.
For the stiff cross-section CS 2, the critical external load q_y up to $\rho \approx 0.80$ rad is the one that plasticizes the bridge's cross-section, while for $\rho > 0.80$ rad the critical load decreases.

6 CONCLUSIONS

On the basis of the chosen bridge models, we may draw the following conclusions:

1. A mathematical model for studying the static stability of a cable-stayed bridge with curved-in-plane deck has been presented. Using the formulae of a previous publication of authors and following the classical way of linear theory, the above bridge is studied for two cases: (a) the unloaded bridge and (b) the case of the loaded bridge by a horizontal load applied vertically to the bridge's axis.
2. The unloaded bridge, for any combination of geometric data buckles at $\rho \approx \pi$ (rad).
3. For the case of a bridge loaded by an external load, as in 1(b) is described, we find out that, for the critical loads in relation to angle ρ, two branches appears. The one resembles as an equilibrium path and the other as a stability curve.
4. For small radii, the governing curves are the resembling as equilibrium paths while for greater radii are the resembling as stability curves.

REFERENCES

1. M.C. Troitsky. *Cable-stayed bridges*. Croby Lockwood Staples, London, 1977.
2. I.G. Raftoyiannis, G.T. Michaltsos. Curved-in-plane cable-stayed bridges. A mathematical model *Int. Journal of Structural Stability and Dynamics* (2012) **12**(3).
3. J.M.W. Brownjohn, J. Lee, B. Cheong. Dynamic performance of a curved cable-stayed bridge. *Engineering Structures* (1999) **21** 1015-1027.
4. D.M. Siringorino, Y. Fujino. Dynamic characteristics of a curved cable-stayed bridge identified from strong motion records *Engineering Structures* **29**(8) (2006) 2001-2017.
5. D. Bruno, V. Golotti. Vibration analysis of cable-stayed bridges *Journal of Structures Engineering International* (1994) **1** 23-28.