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MATHEMATICAL MODELING OF RECTANGULAR LAMINATED PLATES IN BENDING

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ABSTRACT: From the point of view of mechanics of engineering materials, the deformation of a plate subjected to transverse loading consists of two components: flexural deformation due to rotation of cross-sections, and shear deformation due to sliding of sections or layers. The resulting deformation depends on two parameters: the thickness to length ratio and the ratio of elastic to shear moduli. When the thickness to length ratio is small, the plate is considered thin, and it deforms mainly by flexure or bending; whereas when the thickness to length and the modular ratios are both large, the plate deforms mainly through shear. Due to the high ratio of in-plane modulus to transverse shear modulus, the shear deformation effects are more pronounced in the composite laminates subjected to transverse loads than in the isotropic plates under similar loading conditions.

Mathematical models for rectangular laminated plates in bending need to determine the real stress-strain state in the laminated plate, which requires the application of more accurate theories. In addition, it is important to find a balance between the desired accuracy and calculation costs.

Different theories for rectangular plate analysis have been reviewed. These theories can be divided into two major categories, the individual layer theories (IL), and the equivalent single layer (ESL) theories. These categories are further divided into sub – theories by the introduction of different assumptions.

KEYWORDS: Microstructure and Macrostructure, Composites, Laminates, Modeling, Theories of plates, Numerical methods

1 INTRODUCTION

Composite structures which are made of layers of two or more materials are called laminated composite plates. Layers are the basic elements of laminated composite plates and they are made of fibers impregnated in suitable matrix materials. The fibers are load bearing elements of the layer, while the matrix protects the fibers from external effects, holds the fibers together and performs uniform distribution of the influences to each of the fibers. The materials used for fibers have better properties and greater capacity compared to the matrix,

and the geometrical characteristics of the fiber cross section are significantly smaller than its length. Materials used for fibers can be Aluminium, copper, iron, nickel, steel, titanium, or organic materials such as glass, carbon, and graphite. A layer with unidirectional fibers has significantly better characteristics in fiber direction than in a direction perpendicular to the fiber.
Heterogeneity of anisotropic laminated composite plates often causes the appearance of a large number of imperfections that can occur in laminated composite plates. General deformation of laminated plates is often defined by complex coupling between the axial deformation, bending, and shear deformation. In laminated composite plates for smaller aspect ratio, the importance of shear deformation is higher than in the corresponding homogeneous isotropic plates.

Mathematical models for these particular problems need to determine the real stress-strain state in the laminated plate, which requires the application of more accurate theories. In addition, it is important to find a balance between the desired accuracy and calculation costs.

Based on different assumptions for displacement fields, different theories for plate analysis have been devised. These theories can be divided into two major categories, the individual layer theories (IL), the equivalent single layer (ESL) theories as mentioned in Marina Rakočević [1], Osama Khayal [2], [3] and [4]. These categories are further divided into sub – theories by the introduction of different assumptions. For example, the second category includes the classical laminated plate theory (CLPT), the first order and higher order shear deformation theories (FSDT and HSDT) as stated in Refs. {[5] – [8]}.

In the individual layer laminate theories, each layer is considered as a separate plate. Since the displacement fields and equilibrium equations are written for each layer, adjacent layers must be matched at each interface by selecting appropriate interfacial conditions for displacements and stresses. In the ESL laminate theories, the stress or the displacement field is expressed as a linear combination of unknown functions and the coordinate along the thickness. If the in – plane displacements are expanded in terms of the thickness co – ordinate up to the n^{th} power, the theory is named n^{th} order shear deformation theory. The simplest ESL laminate theory is the classical laminated plate theory (CLPT). This theory is applicable to homogeneous thin plates (i.e. the length to thickness ratio $a / h > 20$). The classical laminated plate theory (CLPT), which is an extension of the classical plate theory (CPT) applied to laminated plates was the first theory formulated for the analysis of laminated plates by Reissner and Stavsky [9] in 1961, in which the Kirchhoff and Love assumption that normal to the mid – surface before deformation remain straight and normal to the mid – surface after deformation is used, but it is not adequate for the flexural analysis of moderately thick laminates. However, it gives reasonably accurate results for many engineering problems i.e. thin composite plates, as stated by Srinivas and Rao [10], Reissner and Stavsky [9]. This theory ignores the transverse shear stress components and models a laminate as an equivalent single layer. The classical laminated plate theory (CLPT) under – predicts deflections as proved by Turvey and Osman [11], [12], [13] and Reddy [6] due to the neglect of transverse shear strain. The errors in deflection are even higher for plates made of advanced filamentary composite materials like graphite – epoxy and boron – epoxy whose elastic modulus to shear modulus ratios are very large (i.e. of the order of 25 to 40, instead of 2.6 for typical isotropic materials). However, these composites are susceptible to thickness effects because their effective transverse shear moduli are significantly smaller than the effective elastic modulus along the fiber direction. This effect has been confirmed by Pagano [14] who obtained analytical solutions of laminated plates in bending based on the three – dimensional theory of elasticity. He proved that classical laminated plate theory (CLPT) becomes of less accuracy as the side to thickness ratio decreases. In particular, the deflection of a plate predicted by CLPT is considerably smaller than the analytical value for side to thickness ratio less than 10. These high ratios of elastic modulus to shear modulus render classical laminate theory as inadequate for the analysis of composite plates. In the first order shear deformation theory (FSDT), the transverse planes, which are originally normal and straight to the mid – plane of the plate, are assumed to remain straight but not necessarily normal after deformation, and consequently shear correction factors are employed in this theory to adjust the transverse shear stress, which is constant through thickness. Recently Reddy [15] and Phan and Reddy [16] presented refined plate theories that used the idea of expanding displacements in the powers of thickness coordinate. Numerous studies involving the application of the first – order theory to bending, vibration and buckling analyses can be found in the works of Reddy [17], and Reddy and Chao [18].

In order to include the curvature of the normal after deformation, a number of theories known as higher – order shear deformation theories (HSDT) have been devised in which the displacements are assumed quadratic or cubic through the thickness of the plate. In this aspect, a variationally consistent higher – order theory which not only accounts for the shear deformation but also satisfies the zero transverse shear stress conditions on the top and bottom faces of the plate and does not require correction factors was suggested by Reddy [15]. Reddy's modifications consist of a more systematic derivation of displacement field and variationally consistent derivation of the equilibrium equations. The refined laminate plate theory predicts a parabolic distribution of the transverse shear stresses through the thickness, and requires no shear correction coefficients.

There are two main theories of laminated plates depending on the magnitude of deformation resulting from loading a plate and these are known as the linear and nonlinear theories of plates. The difference between the two theories is that the deformations are small in the linear theory, whereas they are finite or large in the nonlinear theory.

2 LINEAR THEORY OF RECTANGULAR LAMINATED PLATES

2.1 Assumptions

1- The plate shown in Figure 1 is constructed of an arbitrary number of orthotropic layers bonded together as in Figure 2. However, the orthotropic axes of material symmetry of an individual layer need not coincide with the axes of the plate.
2- The displacements u, v and w are small compared to the plate thickness.
3- In-plane displacements u and v are linear functions of the z-coordinate.
4- Each ply obeys Hook's law.
5- The plate is flat and has constant thickness.
6- There are no body forces such as gravity force.
7- The transverse normal stress is small compared with the other stresses and is therefore neglected.

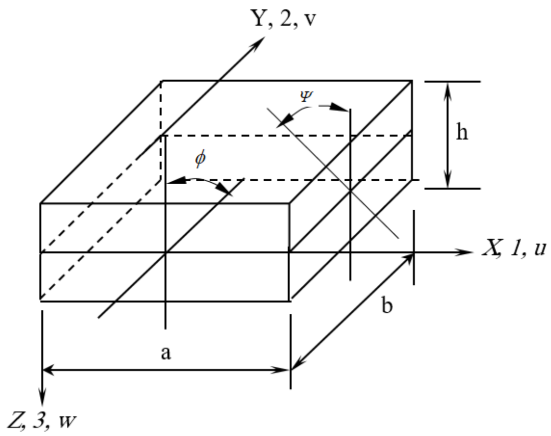

Figure 1. A plate showing dimensions and deformations

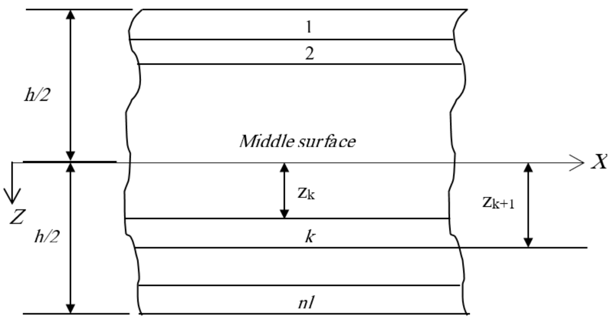

Figure 2. Geometry of an *n* -layered laminate

2.2 Equations of equilibrium

The stresses within a body vary from point to point. The equations governing the distribution of the stresses are known as the equations of equilibrium. Consider the static equilibrium state of an infinitesimal parallel piped with surfaces parallel to the co-ordinate planes. The resultants stresses acting on the various surfaces are shown in Figure 3. Equilibrium of the body requires the vanishing of the resultant forces and moments.

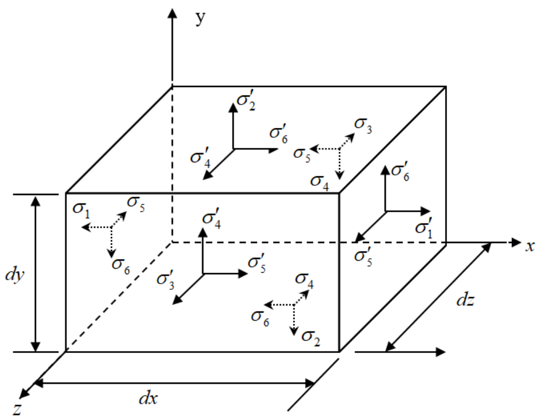

Figure 3. Stresses acting on an infinitesimal element

Where the dash indicates a small increment of stress e.g. $\sigma_1' = \sigma_1 + \frac{\partial \sigma_1}{\partial x} dx$

The forces in the direction of *x* are shown in Figure 4. The sum of these forces gives the following equation.

$$\frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_6}{\partial y} + \frac{\partial \sigma_5}{\partial z} = 0 \quad (1)$$

By summing forces in the directions y and z, the following two equations are obtained:

$$\frac{\partial \sigma_6}{\partial x} + \frac{\partial \sigma_2}{\partial y} + \frac{\partial \sigma_4}{\partial z} = 0 \quad (2)$$

$$\frac{\partial \sigma_5}{\partial x} + \frac{\partial \sigma_4}{\partial y} + \frac{\partial \sigma_3}{\partial z} = 0 \quad (3)$$

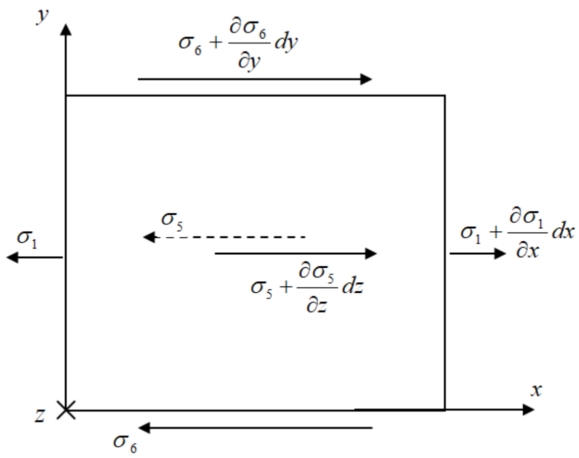

Figure 4. Stresses acting in the *x*–direction

In order to facilitate the analysis of a multi-layered plate as a single layer plate, stress resultants and stress couples are introduced and defined as follows:

$$[N_i, M_i] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \sigma_i [1, z] dz \quad (i = 1,2,6) \tag{4}$$

$$[Q_1, Q_2] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} (\sigma_5, \sigma_4) dz \tag{5}$$

Where z_k and z_{k+1} are the distances of top and bottom surfaces of the k^{th} ply from the middle plane of the plate as shown in Figure 2. The stress resultants and stress couples are clearly shown in Figure 5 and 6 respectively.

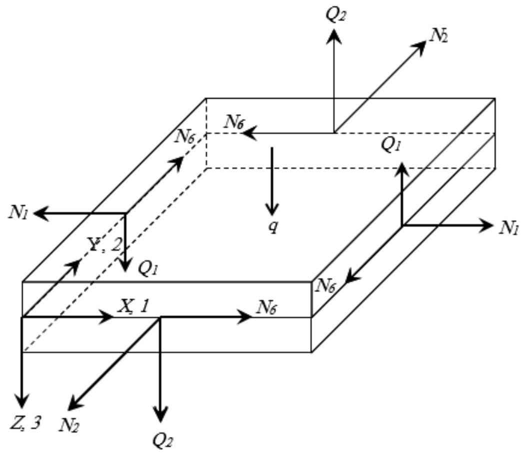

Figure 5. Nomenclature for stress resultants

When integrating equation (1) term by term across each ply, and summing over the plate thickness, it will be converted to:

$$\sum_{k=1}^{n}\int_{Z_k}^{Z_{k+1}}\frac{\partial\sigma_1}{\partial x}dz+\sum_{k=1}^{n}\int_{Z_k}^{Z_{k+1}}\frac{\partial\sigma_6}{\partial y}dz+\sum_{k=1}^{n}\int_{Z_k}^{Z_{k+1}}\frac{\partial\sigma_5}{\partial z}dz=0$$

In order to introduce the stress resultants given in equation (4), summation can be interchanged with differentiation in the first two terms.

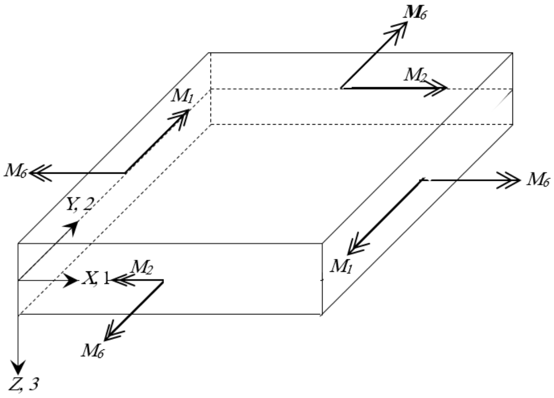

Figure 6. Nomenclature for stress couples

$$\frac{\partial}{\partial x}\left[\sum_{k=1}^{n}\int_{Z_k}^{Z_{k+1}}\sigma_1 dz\right]+\frac{\partial}{\partial y}\left[\sum_{k=1}^{n}\int_{Z_k}^{Z_{k+1}}\sigma_6 dz\right]+\left[\sum_{k=1}^{n}\sigma_5\right]_{Z_k}^{Z_{k+1}}=0$$

The first and second bracketed terms, according to equation (4), are N_1 and N_6 receptively. The last term must vanish because between all plies the inter-laminar shear stresses cancel each other out, and the top and bottom surfaces of the plate are assumed shear stress free.

The first integrated equation of equilibrium can then be written in the following form:

$$\frac{\partial N_1}{\partial x}+\frac{\partial N_6}{\partial y}=0 \tag{6}$$

Similarly equations (2) and (3) can be integrated to give:

$$\frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}=0 \tag{7}$$

$$\frac{\partial Q_1}{\partial x}+\frac{\partial Q_2}{\partial y}+\mathrm{q}=0 \tag{8}$$

where: $q = \sigma_3\left(-\frac{h}{2}\right) - \sigma_3\left(\frac{h}{2}\right)$

The equations of moment equilibrium can be obtained by multiplying equation (1) by *z* and integrating with respect to *z* over plate thickness which yields the following equation:

$$\sum_{k=1}^{n}\int_{z_k}^{Z_{k+1}} \frac{\partial\sigma_1}{\partial x} zdz + \sum_{k=1}^{n}\int_{z_k}^{Z_{k+1}} \frac{\partial\sigma_6}{\partial y} zdz + \sum_{k=1}^{n}\int_{z_k}^{Z_{k+1}} \frac{\partial\sigma_5}{\partial z} zdz = 0$$

When integration and summation are interchanged with differentiation and the stress couples given in equation (4) are introduced, the first two terms become $\left(\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y}\right)$. The third term must be integrated by parts as follows:

$$\sum_{k=1}^{n}\int_{z_k}^{Z_{k+1}} \frac{\partial\sigma_5}{\partial z} zdz = \sum_{k=1}^{n}\int_{z_k}^{Z_{k+1}} \left\{\left[z\sigma_5\right]_{Z_k}^{Z_{k+1}} - \int_{z_k}^{Z_{k+1}} \sigma_5 dz\right\}$$

The first term on the right hand side of the above equation represents the moments of all inter-lamina stresses between plies which again must cancel each other out. The last term, according to equation (5), is – Q_1. Hence the integrated moment equilibrium equation is:

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 = 0 \qquad (9)$$

When equation (2) is treated similarly, it yields the following equation:

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 = 0 \qquad (10)$$

Hence, the equilibrium equations of the plate are the five equations, i.e. equations (6) to (10).

2.3 The strain-displacement equations

Figure 7 shows a small element *ABCD* in the Cartesian co-ordinates *x, y* which deforms to $A'B'C'D'$. The deformations can be described in terms of extensions of lines and distortion of angles between lines. From Figure 7, it is possible to write expressions for linear and shear strains as follows:

$$\varepsilon_1 = \left\{\frac{\left[u + (\partial u/\partial x)dx\right] - u}{dx}\right\} = \frac{\partial u}{\partial x} \qquad (11)$$

Similarly:

$$\varepsilon_2 = \frac{\partial v}{\partial y} \qquad (12)$$

If θ is very small, then,

$$\theta_x = tan\theta_x = \frac{\partial v}{\partial x}$$

$$\theta_y = tan\theta_y = \frac{\partial u}{\partial y}$$

Hence, the shear strain which is the change in the right angle ∟ *BAD* is:

$$\varepsilon_6 = \theta_x + \theta_y = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{13}$$

For a three-dimensional problem, the following strains may be added:

$$\varepsilon_3 = \frac{\partial w}{\partial z} \tag{14}$$

$$\varepsilon_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{15}$$

$$\varepsilon_5 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{16}$$

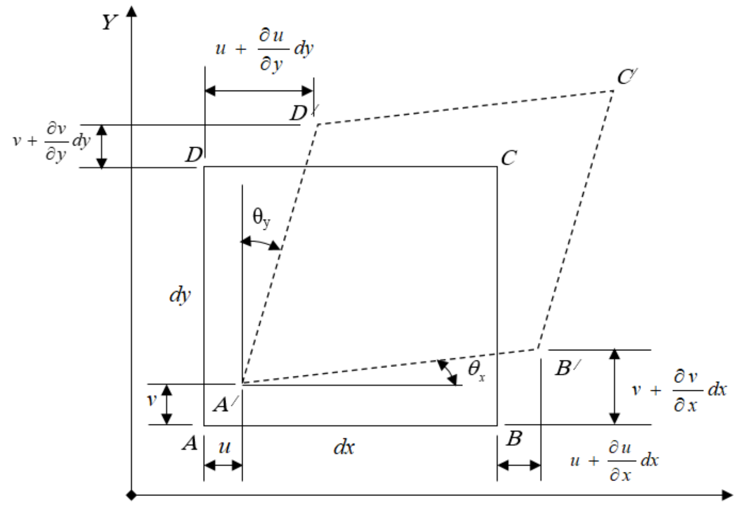

Figure 7. Small element *ABCD* in the artesian co-ordinates *x, y*

The displacements, which comply with assumption (3), are:

$$u = u^o(x,y) + z\Phi^o(x,y)$$
$$v = v^o(x,y) + z\Psi^o(x,y) \qquad (17)$$
$$w = w^o(x,y)$$

Where u^o ,v^o ,and w^o are the displacements of the middle surface of the plate. When equation (17) is differentiated and substituted in Equations (11 –16), the following strain displacement relations are obtained.

$$\varepsilon_1 = \frac{\partial u^o}{\partial x} + z\frac{\partial \Phi}{\partial x}$$
$$\varepsilon_2 = \frac{\partial v^o}{\partial y} + z\frac{\partial \Psi}{\partial y}$$
$$\varepsilon_6 = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} + z\left\{\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x}\right\} \qquad (18)$$
$$\varepsilon_4 = \frac{\partial w}{\partial y} + \Psi$$
$$\varepsilon_5 = \frac{\partial w}{\partial x} + \Phi$$

2.4 The constitutive equations

The constitutive equations of an individual lamina, k, are of the form:

$$\sigma_i^{(k)} = C_{ij}^{(k)}\varepsilon_j^{(k)} \qquad (i, j = 1,2,6) \qquad (19)$$

Where $\sigma_i^{(k)}$ and $\varepsilon_j^{(k)}$ are the stresses and strains in the lamina referred to the plate axes .Using equation (18) in the form $\varepsilon_i = \varepsilon_i^o + z\chi_i^o$ (i =*1, 2, 6*)

$$where\, \varepsilon_1^o = \frac{\partial u^o}{\partial x} \;,\; \varepsilon_2^0 = \frac{\partial v^o}{\partial y} \;,\; \varepsilon_6^o = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x}$$
$$\chi_1^o = \frac{\partial \Phi}{\partial x} \;,\; \chi_2^o = \frac{\partial \Psi}{\partial y} \;,\; \chi_6^o = \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x}$$

Then, equation (19) becomes:-

$$\sigma_i = C_{ij}\left(\varepsilon_j^o + z\chi_j^o\right) \qquad (20)$$

Substitute equation (20) in equation (4) to give:

$$N_i = \sum_{k=1}^{nl} \int_{z_k}^{z_{k+1}} C_{ij}\left(\varepsilon_j^o + z\chi_j^o\right)dz \qquad (21)$$

Equation (21) can be written in the form:

$$N_i = A_{ij}\varepsilon_j^o + B_{ij}\chi_j^o \qquad (i = 1,2,6) \tag{22}$$

Similarly using equation (20) in equation (4) gives:

$$M_i = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} C_{ij}\left(\varepsilon_j^o + z\chi_j^o\right) z\, dz \tag{23}$$

Equation (23) can be written in the form:

$$M_i = B_{ij}\varepsilon_j^o + D_{ij}\chi_j^o \qquad (i = 1,2,6) \tag{24}$$

Where *Aij*, *Bij*, and Dij, (*i*, *j*=1, *2*, *3*) are respectively the membrane rigidities, coupling rigidities and flexural rigidities of the plate. The rigidities Bij display coupling between transverse bending and in-plane stretching. The coupling will disappear when the reference plane is taken as the plate mid-plane for symmetric laminate .The rigidities are calculated as follows:

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} C_{ij}\left(1, z, z^2\right) dz \tag{25}$$

Hence, the laminate constitutive equations can be represented in the form:

$$\begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{Bmatrix} \varepsilon_j^o \\ \chi_j^o \end{Bmatrix} \tag{26}$$

$$\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{27}$$

Where A_{ij} (*i*, *j* =4, 5) denote the stiffness coefficients, and are calculated as follows:

$$A_{ij} = \sum_{k=1}^{n} K_i K_j \int_{z_k}^{z_{k+1}} C_{ij}^{(k)} dz\,, \; (i, j = 4,5) \tag{28}$$

Where K_i ,K_j are the shear correction factors.

2.5 Boundary conditions

The proper boundary conditions are those which are sufficient to guarantee a unique solution of the governing equations. To achieve that goal, one term of each of the following five pairs must be prescribed along the boundary.

$$N_n \;\; or \;\; u_n \;\; ; N_{ns} \;\; or \;\; u_s \; ; M_n \;\; or \;\; \phi_n \; ; M_s \;\; or \;\; \phi_s \;\; ; Q \;\; or \;\; w \tag{29}$$

Where the subscripts *n* and *s* indicate the normal and tangential directions respectively. The boundary conditions used in this thesis are given in Appendix C.

3 NONLINEAR THEORY OF RECTANGULAR LAMINATED PLATES

3.1 Assumptions

The assumptions made in the nonlinear theory of laminated plates are the same as those listed for linear analysis, section 2.1, except for assumption (2), which is concerned with the magnitude of deformations. In the nonlinear theory, in-plane displacements are again small compared to the thickness of the plate, but the out-of-plane displacement is not.

3.2 Equations of equilibrium

The derivation of the equilibrium equations for finite deformations could be found in references [3], [6], [7] and [8] and which could be written in the following form:

$$\frac{\partial}{\partial x}\left[\sigma_1\left(1+\frac{\partial u}{\partial x}\right)+\sigma_6\frac{\partial u}{\partial y}+\sigma_5\frac{\partial u}{\partial z}\right]+\frac{\partial}{\partial y}\left[\sigma_6\left(1+\frac{\partial u}{\partial x}\right)+\sigma_2\frac{\partial u}{\partial y}+\sigma_4\frac{\partial u}{\partial z}\right]$$
$$+\frac{\partial}{\partial z}\left[\sigma_5\left(1+\frac{\partial u}{\partial x}\right)+\sigma_4\frac{\partial u}{\partial y}+\sigma_3\frac{\partial u}{\partial z}\right]=0 \qquad (30)$$

$$\frac{\partial}{\partial x}\left[\sigma_1\frac{\partial v}{\partial x}+\sigma_6\left(1+\frac{\partial v}{\partial y}\right)+\sigma_5\frac{\partial v}{\partial z}\right]+\frac{\partial}{\partial y}\left[\sigma_6\frac{\partial v}{\partial x}+\sigma_2\left(1+\frac{\partial v}{\partial y}\right)+\sigma_4\frac{\partial v}{\partial z}\right]$$
$$+\frac{\partial}{\partial z}\left[\sigma_5\frac{\partial v}{\partial x}+\sigma_4\left(1+\frac{\partial v}{\partial y}\right)+\sigma_3\frac{\partial v}{\partial z}\right]=0 \qquad (31)$$

$$\frac{\partial}{\partial x}\left[\sigma_1\frac{\partial w}{\partial x}+\sigma_6\frac{\partial w}{\partial y}+\sigma_5\left(1+\frac{\partial w}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\sigma_6\frac{\partial w}{\partial x}+\sigma_2\frac{\partial w}{\partial y}+\sigma_4\left(1+\frac{\partial w}{\partial z}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\sigma_5\frac{\partial w}{\partial x}+\sigma_4\frac{\partial w}{\partial y}+\sigma_3\left(1+\frac{\partial w}{\partial z}\right)\right]=0 \qquad (32)$$

The equilibrium equations of a body undergoing large deformations are given in equations (30) – (32). Assuming that the in-plane displacement gradients are small compared to unity and neglecting the transverse normal stress σ_3, equations (30) – (32) can be written in a simpler form as follows:

$$\frac{\partial\sigma_1}{\partial x}+\frac{\partial\sigma_6}{\partial y}+\frac{\partial\sigma_5}{\partial z}=0 \qquad (33)$$

$$\frac{\partial\sigma_6}{\partial x}+\frac{\partial\sigma_2}{\partial y}+\frac{\partial\sigma_4}{\partial z}=0 \qquad (34)$$

$$\frac{\partial}{\partial x}\left[\sigma_5+\sigma_1\frac{\partial w}{\partial x}+\sigma_6\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial y}\left[\sigma_4+\sigma_6\frac{\partial w}{\partial x}+\sigma_2\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[\sigma_5\frac{\partial w}{\partial x}+\sigma_4\frac{\partial w}{\partial y}+\sigma_3\right]=0 \qquad (35)$$

Integrating equations (33) and (34) over the thickness of the plate as in section 2.2 gives equations (6) and (7) as before. When equation (35) is integrated, it gives:

$$\frac{\partial}{\partial x}\left[Q_1+N_1\frac{\partial w}{\partial x}+N_6\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial y}\left[Q_2+N_6\frac{\partial w}{\partial x}+N_2\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[Q_1\frac{\partial w}{\partial x}+Q_2\frac{\partial w}{\partial y}\right]+q=0 \qquad (36)$$

This can be rewritten in the following form:

$$N_1\frac{\partial^2 w}{\partial x^2}+2N_6\frac{\partial^2 w}{\partial x\partial y}+N_2\frac{\partial^2 w}{\partial y^2}+\frac{\partial Q_1}{\partial x}+\frac{\partial Q_2}{\partial y}+q+\frac{\partial w}{\partial x}\left(\frac{\partial N_1}{\partial x}+\frac{\partial N_6}{\partial y}\right)+\frac{\partial w}{\partial y}\left(\frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}\right)=0 \qquad (37)$$

where: $q=\sigma_3\left(-\frac{h}{2}\right)-\sigma_3\left(\frac{h}{2}\right)$

However, similar to equations (6) and (7), the last two terms in equation (37) must be zero, and so the above equation reduces to:

$$N_1\frac{\partial^2 w}{\partial x^2}+2N_6\frac{\partial^2 w}{\partial x\partial y}+N_2\frac{\partial^2 w}{\partial y^2}+\frac{\partial Q_1}{\partial x}+\frac{\partial Q_2}{\partial y}+q=0 \qquad (38)$$

Multiplying equations (33) and (34) by z and again integrating over the thickness of the plate to obtain equations (9) and (10).

Hence, the governing equations of the plate are the following five equations (6), (7), (38), (9), and (10). It should be noted that the shear deformation theory derived above reduces to classical laminated theory when the transverse shear strains are eliminated by setting:

$$\phi=-\frac{\partial w}{\partial x} \quad ,and \quad \psi=-\frac{\partial w}{\partial y}$$

3.3 The strain-displacement equations

The in-plane displacements *u* and *v* are small, whereas the deflection *w* is of the order of half the plate thickness or more. This assumption implies that:

$$\frac{\partial(u,v)}{\partial(x,y)} << \frac{\partial w}{\partial(x,y)} \qquad (39)$$

Consequently, the expressions for finite strains can be simplified as follows:

$$\begin{aligned}
\varepsilon_1 &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_2 &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\varepsilon_3 &= 0 \\
\varepsilon_4 &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\
\varepsilon_5 &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\
\varepsilon_6 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}.\frac{\partial w}{\partial y}
\end{aligned} \tag{40}$$

The in-plane displacements are again assumed to vary linearly through the thickness of the plate as described for linear analysis i.e.:-

$$\begin{aligned}
u &= u^o(x,y) + z\phi^o(x.y) \\
v &= v^o(x,y) + z\psi^o(x.y) \\
w &= w^o(x,y)
\end{aligned} \tag{41}$$

When these displacements are substituted into equation (40), the following relations are obtained:

$$\begin{aligned}
\varepsilon_i &= \varepsilon_i^o + z\chi_i^o \qquad (i = 1,2,6) \\
\varepsilon_4 &= \frac{\partial w}{\partial y} + \psi \\
\varepsilon_5 &= \frac{\partial w}{\partial x} + \phi
\end{aligned} \tag{42}$$

where:

$$\begin{aligned}
\varepsilon_1^o &= \frac{\partial u^o}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_2^o &= \frac{\partial v^o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2
\end{aligned}$$

$$\varepsilon_6^o = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$

$$\chi_1^o = \frac{\partial \phi}{\partial x}$$

$$\chi_2^o = \frac{\partial \psi}{\partial y}$$

$$\chi_6^o = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}$$

$$\varepsilon_1 = \frac{\partial u^o}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi}{\partial x}$$

$$\varepsilon_2 = \frac{\partial v^o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \psi}{\partial y}$$

$$\varepsilon_6 = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + z\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) \qquad (43)$$

$$\varepsilon_4 = \frac{\partial w}{\partial y} + \psi$$

$$\varepsilon_5 = \frac{\partial w}{\partial x} + \phi$$

3.4 The constitutive equations

These are the same as equations (26), and (27) of section 2.4.

3.5 Boundary conditions

These are the same as equation (29) of section 2.5.

4 TRANSFORMATION EQUATIONS

4.1 Stress-strain equations

For linear elastic materials, the relation between the stress and strain is as follows:

$$\sigma_i' = C_{ij}' \varepsilon_j' \qquad (i, j = 1, 2, \ldots, 6) \qquad (44)$$

Where the first subscript i refers to the direction of the normal to the face on which the stress component acts, and the second subscript j corresponds to the direction of the stress.

When an orthotropic body is in a state of plane stress, the non-zero components of the stiffness tensors C'_{ij} are:

$$C'_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}$$

$$C'_{12} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}$$

$$C'_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

$$C'_{44} = G_{23}, \quad C'_{55} = G_{13}, \quad C'_{66} = G_{12} \tag{45}$$

Where E_1 and E_2 are Young's moduli in directions 1 and 2 respectively. ν_{ij} Is Poisson's ratio of transverse strain in the j-direction when stressed in the *i*-direction (i.e., $\nu_{ij} = -\varepsilon'_j / \varepsilon_i$ When $\sigma'_i = \sigma$ and all other stresses are zero).

4.2 Transformation of stresses and strains

Consider a co-ordinate system rotated anticlockwise through an angle θ, the rotated axes are denoted by $1'$, $2'$ as in Figure 8 below. Consider the equilibrium of the small element ABC shown. Resolving forces parallel to $1'$ axis gives:

$$\sigma'_{11}ds = \sigma_{11}dy\cos\theta + \sigma_{22}dx\sin\theta + \sigma_{12}dx\cos\theta + \sigma_{12}dy\sin\theta \tag{46}$$

On rearranging, the expression reduces to:

$$\sigma'_{11} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\sin\theta\cos\theta + \sigma_{12}dy\sin\theta \tag{47}$$

Resolving forces parallel to $2'$ axis gives:

$$\sigma'_{12}ds = -\sigma_{11}dy\sin\theta + \sigma_{22}dx\cos\theta + \sigma_{12}dy\cos\theta - \sigma_{12}dx\sin\theta \tag{48}$$

This can then be written in the form:

$$\sigma'_{12} = -\sigma_{11}\sin\theta\cos\theta + \sigma_{22}\sin\theta\cos\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right) \tag{49}$$

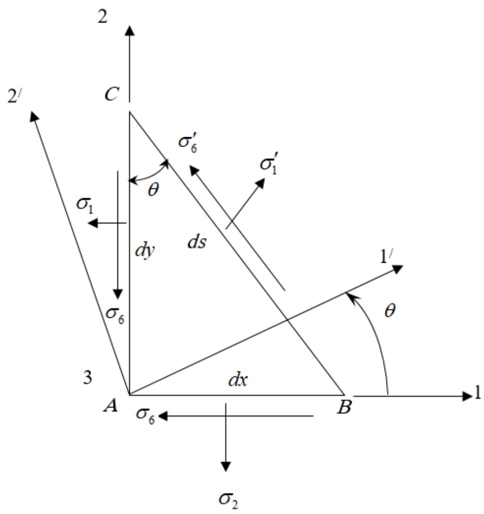

Figure 8. Stresses on a triangular element

The same procedure is applied to obtain the other transformed stresses which may be written in a matrix form as:

$$\{\sigma_i'\} = [M]\{\sigma_i\} \tag{50}$$

Where:

$$[M] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix}$$

The strains are transformed similarly:

$$\{\varepsilon_i'\} = [N]\{\varepsilon_j\} \tag{51}$$

Where:

$$[N] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix}$$

4.3 Transformation of the elastic moduli

In general the principle material axes ($1^{/}$, $2^{/}$, $3^{/}$) are not aligned with the geometric axes (1, 2, 3) as shown in Figure 9 for a unidirectional continuous fiber composite. It is necessary to be able to relate the stresses and strains in both co-ordinate systems. This is achieved by multiplying equation (50) by $[M]^{-1}$ i.e.:

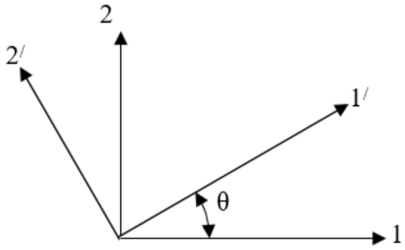

Figure 9. A generally orthotropic plate

$$\{\sigma_i\} = [M]^{-1}\{\sigma_i'\} \tag{52}$$

Substitute equation (44) in equation (52) to obtain:

$$\{\sigma_i\} = [M]^{-1}[C_{ij}']\{\varepsilon_j'\} \tag{53}$$

Then, substitute Eqn. (2.51) in Eqn. (2.53)

$$\{\sigma_i\} = [M]^{-1}[C_{ij}'][N]\{\varepsilon_j\} \tag{54}$$

This equation can be written as:

$$\{\sigma_i\} = \lfloor C_{ij} \rfloor \{\varepsilon_j\} \tag{55}$$

Where $\{C_{ij}\} = [M]^{-1}[N][C_{ij}']$

Equation (55) gives the constitutive equation for a generally orthotropic lamina in which the material axes and geometric axes are not aligned. The constants C_{ij} are as follows:

$$C_{11} = C_{11}'m^4 + 2m^2n^2(C_{12}' + 2C_{66}') - 4mn(C_{16}'m^2 + C_{26}'n^2) + C_{22}'n^4$$

$$C_{12} = m^2n^2(C_{11}' + C_{22}' - 4C_{66}') + 2mn(m^2 - n^2)(C_{16}' + C_{26}') + (m^4 + n^4)C_{12}'$$

$$C_{13} = C_{13}'m^2 + C_{23}'n^2$$

$$C_{16} = m^2n^2[C_{11}'m^2 - C_{22}'n^2 - (C_{12}' + 2C_{66}')(m^2 - n^2)] + m^2(m^2 - 3n^2)C_{16}' + n^2(3m^2 - n^2)C_{26}'$$

$$C_{22} = C_{11}'n^4 + 2m^2n^2(C_{12}' + 2C_{66}') + 4mn(C_{26}'m^2 + C_{16}'n^2) + C_{22}'m^4$$

$$C_{23} = C'_{13}n^2 + C'_{23}m^2$$
$$C_{26} = m^2\left(m^2 - 3n^2\right)C'_{26} + mn\left[C'_{11}n^2 - C'_{22}m^2 + \left(C'_{12} + 2C'_{66}\right)\left(m^2 - n^2\right)\right] + n^2\left(3m^2 - n^2\right)C'_{16}$$
$$C_{33} = C'_{33}$$
$$C_{36} = \left(C'_{23} - C'_{13}\right)mn$$
$$C_{44} = C'_{44}m^2 + 2mnC'_{45} + C'_{55}n^2$$
$$C_{45} = \left(m^2 - n^2\right)C'_{45} - mn\left(C'_{44} - C'_{55}\right)$$
$$C_{55} = C'_{55}m^2 - 2mnC'_{45} + C'_{44}n^2$$
$$C_{66} = m^2n^2\left(C'_{11} + C'_{22} - 2C'_{12}\right) - 2mn\left(m^2 - n^2\right)\left(C'_{26} - C'_{16}\right) + \left(m^2 - n^2\right)^2 C'_{66}$$
$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0 \qquad (56)$$

Where $m=\cos\theta$ and $n=\sin\theta$

5 CONCLUSIONS

There are two main theories of laminated plates depending on the magnitude of deformation resulting from loading a plate and these are known as the linear and nonlinear theories (i.e. small and large deformation theories) of plates. The difference between the two theories is that the deformations are small in the linear theory, whereas they are finite or large in the nonlinear theory.

Mathematical models for these particular problems need to determine the displacement equations and consequently the real stress - strain state in the laminated plate, which requires the application of more accurate theories. In addition, it is important to find a balance between the desired accuracy and calculation costs.

Based on different assumptions for displacement fields, different theories for plate analysis have been reviewed. These theories can be divided into two major categories, the individual layer theories (IL), and the equivalent single layer (ESL) theories. These categories are further divided into sub – theories by the introduction of different assumptions. For example, the second category includes the classical laminated plate theory (CLPT), the first order and higher order shear deformation theories (FSDT and HSDT).

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