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DIRECT MOVING LOAD ENVELOPE ANALYSIS EQUATIONS FOR BRIDGE GIRDERS

Daud Abdoh

City University of Hong Kong, Department of Architecture and Civil Engineering, Hong Kong, China
e-mail: dabdoh2-c@my.cityu.edu.hk

ABSTRACT: Accurate and practical direct envelope equations for moving load analysis are needed these days for many reasons. Over or under estimating the load capacity in pre-stressed bridge girders may raise concerns about safety and serviceability. Moreover, direct envelope equations lead to efficient and economical design of bridge structures. Finally, these equations can provide more accurate results compared to those obtained using a commercial software program.

This study develops direct envelope analysis equations for moving loads on bridge girders. The analysis consists of maximum positive support moments, positive and negative shear envelope equations, negative moment envelope equations and maximum point of deflection. The analysis is based on one dimensional slope moment matrix method designed to calculate support moments of structural members. Maximum point of deflection is based on general deflection equation. Fixed and pinned end support conditions are applied since there is no cantilever bridge girder. Detailed Comparison between this paper results and commercial software program (STAAD) will be made to prove their accuracy and practicality.

Most of the commercial software programs have been concentrated on interval distance calculation of moving load analysis. In comparison, this study is conducted on the continuous moving of wheel loads (interval distance is equal zero); hence the calculations obtained are more accurate.

KEYWORDS: Bridge Girder; Equations; General Analysis; Envelope; Moving loads.

1. INTRODUCTION

Moving load envelope analysis is considered to be challenging topic especially in bridge engineering, several research projects have been published in that filed, and some of them trying to solve the problem by using numerical methods like finite element analysis by setting order differential equations for the interaction between beam and moving dynamic system [3] or using the spatial discretization of the governing partial differential equation of the vehicle-bridge [6]. Others have used force based finite elements and employ new numerical integration approach [4].

Using vibration techniques for moving load envelope analysis equations were used also like modified beam vibration functions as the assumed modes [5]. In general, previous researches are based on numerical methods (approximation).

The author published his paper about deriving envelope analysis equations for two spans bridge girder on 2015 [1], he has addressed the direct envelope equations for two spans bridge girders based on direct integration method (analytical approach), pinned end boundary conditions were taken only. He has addressed in that paper the new challenges like expanding the envelope equations for multiple spans for different boundary conditions. Later he has focused on developing new method and technique for one dimensional structural analysis. The most traditional structural analysis methods are moment distribution method and stiffness matrix method, due to need for more efficient and practical method in structural analysis, he has published another paper [2] giving the entrance for this current research and to solve the future challenges which already have been addressed in the first published paper [1].

Derivation of general analysis envelope equations based on slop moment matrix method which is 1-D structural analysis method designed to calculate support moments of structural members for any loading patterns, any boundary conditions and any number of spans. Fixed and pinned support conditions were taken for bridge girders since there is no cantilever bridge girder.

Slop moment matrix method is presented first for concentrated load pattern; connection between slop moment matrix method and general analysis envelope equations was established clearly, the following sections were presented in order: Maximum negative support moments, Envelope positive and negative shear equations, envelope negative moment equations, and maximum point of deflection.

Envelope Deflection equation and maximum point of deflection were derived based on general deflection equation, all equations and derivation were made for any selected span from multiple bridge girder spans, truck or vehicle loads were considered to act on each span together, thus equations are established for each span separately, combining these equations will form the total bridge girder envelope equations.

These derived equations are powerful because there are no limitations on number of spans, number of axial wheel loads, type of vehicle or truck load, boundary conditions, nor on geometry of the bridge (span lengths) to give structural and bridge engineers the most effective direct equations for bridge girder analysis without using any software. However, bridge engineering programmers can use these direct equations in their codes which in turn can minimize the programming cost and time, also these equations are applied for any type of moving load analysis if same assumptions assumed in this research are taken. An Example was taken from STAAD software to compare the results with these new equations.

2. SIGN CONVENTION

Load downward was assumed to be positive, load upward was negative, positive shear was upward, negative shear was downward, positive moment was concave down (tension at top), negative moment was concave up (compression at top), positive slope for counterclockwise rotation, negative slope for clockwise rotation, negative deflection was downward, positive deflection was upward.

3. SLOP MOMENT MATRIX METHOD AND GENERAL ANALYSIS ENVELOPE EQUATIONS

Slop moment matrix method is direct structural analysis method by which support moments can be calculated by forming two matrices, square span lengths matrix [R] and one column matrix of loading factors as follows:

$$[M]= [R]^{-1} \times [N] \tag{1}$$

$$\begin{pmatrix} M_1 \\ M_2 \\ \cdots \\ \cdots \\ M_n \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} 2L_1 & L_1 & 0 & 0 & 0 & 0 \\ L_1 & 2(L_1+L_2) & L_2 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & L_{n-1} & 2(L_{n-1}+L_n) & L_n \\ 0 & 0 & 0 & 0 & L_n & 2L_n \end{pmatrix}^{-1} \begin{pmatrix} M_1 \\ M_2 \\ \cdots \\ \cdots \\ M_n \\ M_{n+1} \end{pmatrix}$$

Where:

[m]: Is the support moments matrix, Unknowns

[R]: Is the span lengths matrix.

[N]: Is the matrix of loading factors around each support, which is simply the contribution of loading to rotation or slop (θ) of member at the support.

n: number of spans.

L_1, L_2 , L_n, L_{n-1} = lengths of span 1, 2, n, n-1

If first support is pinned, then there is no moment at first support, M_1 and N_1 will be zero, if last support is pinned then M_{n+1} and N_1 will be zero.

It was noticed from above that loading is separate from geometry (span lengths) and boundary conditions giving the entrance for general analysis envelope equations, since moving loads will change only the loading matrix [N], the only part needed for equations solving is geometry matrix (spans lengths, [R]), solving this matrix is done by finding the matrix inverse $[R]^{-1}$, since moving loads are generally occurs in one span, then two items in loading matrix [N] will have values above zero , remaining values are zero because there are no loadings on other spans.

According to formula number 1, each support moment is calculated by multiplying two constants in $[R]^{-1}$ with corresponding two loading factors in [N] matrix, remaining multiplications are zero, according to following formulas:

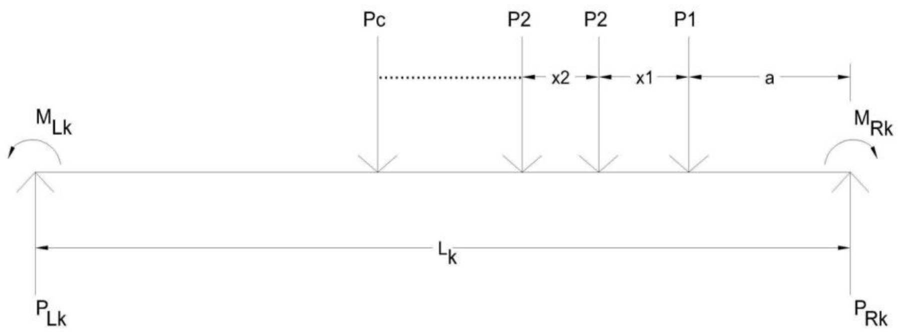

Figure 1. Moving loads on selected span (k)

$$M_{Lk}=R^{-1}_{(k,k)}\times N_k + R^{-1}_{(k,k+1)}\times N_{k+1} \quad (1)$$

$$M_{Rk}=R^{-1}_{(k+1,k)}\times N_k + R^{-1}_{(k+1,k+1)}\times N_{k+1} \quad (2)$$

Where:

$R^{-1}_{(k,k)}$ = Constant element in span lengths matrix inverse at row k and column k.

$R^{-1}_{(k,k+1)}$ = Constant element in span lengths matrix inverse at row k and column k+1.

$R^{-1}_{(k+1,k)}$ = Constant element in span lengths matrix inverse at row k+1 and column k.

$R^{-1}_{(k+1,k+1)}$ = Constant element in span lengths matrix inverse at row k+1 and column k+1.

M_{Lk} = Left support moment for span k

M_{Rk} = Right support moment for span k

N_k = Loading and support settlement factor in [N] matrix at row k

N_{k+1} = Loading factor in [N] matrix at row k+1

The following are equations to calculate loading element for matrix [N] for concentrated loads (single and multiple moving), for single concentrated load:

$$N_k=\frac{P\times[L^2\times a-a^3]}{L_k} \quad (3)$$

For Single concentrated load:

$$N_{k+1}=\frac{P\times[a^3+2aL^2-3a^2L]}{L_k} \quad (4)$$

For multiple concentrated load:

$$N_k=\sum_{i=1}^{c}\frac{P_i\times[L^2\times(a+x_{i-1})-(a+x_{i-1})^3]}{L_k} \quad (5)$$

For multiple concentrated load:

$$N_{k+1}=\sum_{i=1}^{c}\frac{P_i\times[(a+x_{i-1})^3+2\times(a+x_{i-1})\times L^2-3\times(a+x_{i-1})^2\times L]}{L_k} \quad (6)$$

Where:

c: Total Number of axial moving loads

P: Axial moving load

L_k: Length of span K (required span to conduct analysis on it)

X_1: distance between axial load.

a: is the distance from first moving load P1 to right support as shown in figure 1.

$x_0 = 0$

As shown above, supports moments for any span can be calculated directly as isolated from other spans, since R^{-1} constants are the only connections between spans which are fixed depends only on geometry (span lengths).

4. MAXIMUM POSITIVE SUPPORTS MOMENTS DUE TO MULTIPLE MOVING LOADS

As presented earlier, supports moments for multiple moving loads can be calculated directly using below formulas:

$$M_{Lk}=R^{-1}_{(k,k)}\times\sum_{i=1}^{c}\frac{P_i\times[L^2\times(a+x_{i-1})-(a+x_{i-1})^3]}{L_k}+R^{-1}_{(k,k+1)}\times\sum_{i=1}^{c}\frac{P_i\times[(a+x_{i-1})^3+2\times(a+x_{i-1})\times L^2-3\times(a+x_{i-1})^2\times L]}{L_k} \quad (8)$$

$$M_{Rk}=R^{-1}_{(k+1,k)}\times\sum_{i=1}^{c}\frac{P_i\times[L^2\times(a+x_{i-1})-(a+x_{i-1})^3]}{L_k}+R^{-1}_{(k+1,k+1)}\times\sum_{i=1}^{c}\frac{P_i\times[(a+x_{i-1})^3+2\times(a+x_{i-1})\times L^2-3\times(a+x_{i-1})^2\times L]}{L_k} \quad (9)$$

It is noticed that both equations for M_{Lk} & M_{Rk} are similar, only difference are the R^{-1} constants, to find the location of vehicle or truck that produces maximum positive supports moments, derivatives of previous two equations 8 and 9 according to variable a (distance from first load to right support as defined before) should be taken zero as follows:

$$\frac{dM_{Lk}}{da} = 0 \tag{10}$$

$$\frac{dM_{Lk}}{da} = R^{-1}_{(k,k)} \times \sum_{i=1}^{c} \frac{P_i\,[L^2-3\,(a+x_{i-1})^2]}{L_k} + R^{-1}_{(k,k+1)} \times \sum_{i=1}^{c} \frac{P_i\,[3(a+x_{i-1})^2+2L^2-6(a+x_{i-1})^1 L]}{L_k} = 0 \tag{11}$$

$$\frac{dM_{Lk}}{da} = \sum_{i=1}^{c} \frac{P_i\,R^{-1}_{(k,k)}\left[L^2-3(a+x_{i-1})^2\right]}{L_k} + \sum_{i=1}^{c} \frac{R^{-1}_{(k,k+1)}\left[3(a+x_{i-1})^2+2L^2-6(a+x_{i-1})^1 L\right]}{L_k} = 0 \tag{12}$$

$$\frac{dM_{Lk}}{da} = a^2 \sum_{i=1}^{c} \frac{P_i\,[3\left(R^{-1}_{(k,k+1)}-R^{-1}_{(k,k)}\right)]}{L_k} + a \sum_{i=1}^{c} \frac{P_i\,[6\,x_{i-1}\left(R^{-1}_{(k,k+1)}-R^{-1}_{(k,k)}\right)-6\,L_k\,R^{-1}_{(k,k+1)}]}{L_k} + \sum_{i=1}^{c} \frac{P_i\,[\,R^{-1}_{(k,k)}\left(L_k^2-3\times x_{i-1}^2\right)+R^{-1}_{(k,k+1)}\left(3x_{i-1}^2+2{L_k}^2-6x_{i-1}L_k\right)]}{L_k} = 0 \tag{13}$$

After derivation formula reached final shape, coefficients of square equations are defined clearly, by using square roots equations, variable a_{max} can be found as follows:

$$a_{max(LK)} = \frac{-B-\sqrt{B^2-4\times A\times C}}{2\times A} \tag{14}$$

Where:

$a_{max\,(LK)}$: Distance from first moving load of vehicle or truck to right support of span k that produces the maximum positive moment at left support of same span.

$$A=\sum_{i}^{c}\frac{P_i\ [3\ \left(R^{-1}_{(k,k+1)}-R^{-1}_{(k,k)}\right)]}{L_k} \quad (15)$$

$$B=\sum_{i}^{c}\frac{P_i\ \left[6\ x_{i-1}\ \left(R^{-1}_{(k,k+1)}-R^{-1}_{(k,k)}\right) - 6\ L_k\ R^{-1}_{(k,k+1)}\right]}{L_k} \quad (16)$$

$$C=\sum_{i=1}^{c}\frac{P_i\ [\ R^{-1}_{(k,k)}\ \left(L_k^2-3\ x_{i-1}^2\right)+R^{-1}_{(k,k+1)}\ \left(3\ x_{i-1}^2+2{L_k}^2-\ 6\ x_{i-1}L_k\right)]}{L_k} \quad (17)$$

To find maximum positive support moment, value of $a_{max\ (LK)}$ is substituted in formula number 9 instead of a, same derivation and equations are applicable to M_{Rk} but replacing $R^{-1}_{(k,k)}$ with $R^{-1}_{(k+1,k)}$ and $R^{-1}_{(k,k+1)}$with $R^{-1}_{(k+1,k+1)}$.

5 ENVELOPE POSITIVE AND NEGATIVE SHEAR EQUATIONS

When support moments M_{Lk} & M_{Rk} are calculated according to formulas 9 and 10, unknown support reactions can be found by applying simple equilibrium equations as shown below:

$$P_{Lk}\times L_k+M_{Rk}=M_{Lk}+\sum_{i}^{c}P_i\times(a+x_{i-1}) \quad (18)$$

$$P_{Lk}=\frac{\sum_i^c P_i\times(a+x_{i-1})+M_{Lk}-M_{Rk}}{L_k} \quad (19)$$

$$P_{Rk}=[\sum_{i}^{c}P_i]\ -P_{Lk} \quad (20)$$

At each point on span k, maximum shear occurs on it when any of moving load reaches that point, thus having multiple possibilities of envelope shear equations equal to number of axial moving loads, maximum shear at any point can be caused when one of the axial loads acting on it, due to nature of shear diagram of concentrated loads (straight lines of different magnitude between axial loads), the envelope shear diagram is the maximum values of all possible shear diagrams as follows:

$$V(x)_p=P_{Lk}-\sum_{n+1}^{c}P_n \quad (21)$$

Where:

n is the order or number of shear equation starts from 1 to c (total number of axial moving loads)

$V(x)_p$: Positive shear equation

(a) Variable in formula 20 should be equal to $a=L-X-\sum_1^n x_{i-1}$

The result is multiple positive shear equations equals to c (number of axial moving loads), the maximum of these shear diagrams at each point is the envelope positive shear equation.

Envelope negative shear diagrams can be found by same manner but using P_{Rk} instead of P_{Lk} because maximum shear occurs from right to left support, in contrast of positive shear that occurs from left to right as follows:

$$V(x)_N=P_{Lk}-\sum_i^c P_i+\sum_{i=1}^n P_{i-1} \quad (22)$$

6 ENVELOPE NEGATIVE MOMENT EQUATION

According to shape of moment diagram under concentrated loads effects, the maximum negative moment within span k occurs under one of axial moving loads, same concept of envelope shear equation, thus having multiple possibilities of maximum negative moment equal to number of axial loads, by applying equilibrium equation, the moment of any point at span k equals to:

$$M(x)_N=M_{Lk}-P_{Lk}\times x+\sum_{i=c}^{n+1} P_i\times x_{i-1} \quad (23)$$

Where:

$M(x)_N$: Negative moment equation

The maximum of all these negative moment equations at each point gives the envelope negative moment equation.

7 ENVELOPE DEFLECTION EQUATION AND MAXIMUM DEFLECTION POINT

Envelope deflection equations were derived based on general deflection equation which states that the deflection of any point can be calculated exactly by knowing the moment value at this point and locating two known points of moment and deflection around this point from each side. To simplify deflection calculation, equivalent moving load (Pe) is taken by equaling the moment of all axial loads with moment of equivalent load around certain point, the maximum deflection of each point on any span can be caused when equivalent load occurs before or after that point thus having two possible envelope deflection equations as follows:

$$\Delta_{(X)B}=\frac{\left[2M_{Xk}L_k+M_{Lk}X+M_{Rk}(L_k-X)-C_{(X)L}\right]X(L_k-X)}{6\ EI\ L}+\frac{6EI[\Delta_R X+\Delta_L(L_k-X)]}{6\ EI\ L} \quad (24)$$

$$\Delta_{(X)A}=\frac{\left[2\ M_{Xk}L_k+M_{Lk}\ X+M_{Rk}\ (L_k\text{-}X)\text{-}\ C_{(X)R}\right]X\ (L_k\text{-}X)}{6\ EI\ L} \qquad (25)$$

Where:

X: Distance from left support to required point

$\Delta_{(X)B}$: Deflection at point X when equivalent load is moving before that point.

$\Delta_{(X)A}$: Deflection at point X when equivalent load is moving after that point.

M_{Xk} : Moment at required point X starts from left support of span k

M_{Lk} & M_{Rk} : Left and right support Moments of span k as defined before.

L_K : Length of span k.

$C_{(X)L}$, $C_{(X)R}$: are loading coefficient at point X

$$C_{(X)L}=\frac{P_e\times[(L_k\text{-}a\text{-}x)^3+2\times(L_k\text{-}a\text{-}x)\times L_k^2\text{-}3\times a(L_k\text{-}a\text{-}x)^2\times L_k]}{L_k}$$

$$C_{(X)R}=\frac{P_e\times[L^2\times(L_k\text{-}a\text{-}x)\text{-}(L_k\text{-}a\text{-}x)^3]}{L_k}$$

It is noticed that both loading coefficient $C_{(X)L}$, $C_{(X)R}$ are similar to N_{k+1} and N_k as defined in formulas 4 and 5, except that a is replaced with L_k-a-x and P with P_e.

As seen from previous calculation, it is very hard and not friendly so rather than deriving envelope deflection equation, maximum point of deflection will be located, the maximum point of deflection will always be defined when equivalent moving load (Pe) acts on it, Excel sheet is used to calculate deflection under moving load, effects of two corresponding spans were taken before and after required span, other far spans have neglect effect on maximum point deflection location, main parameters were the ration of span length left over required span length ($\frac{L_{k-1}}{L_k}$) and ration of right span length over required span length ($\frac{L_{k+1}}{L_k}$), at each ration maximum deflection point is located, the results were sets of ratio values for maximum point of deflection ($\frac{a}{L_k}$) as shown in figure 2:

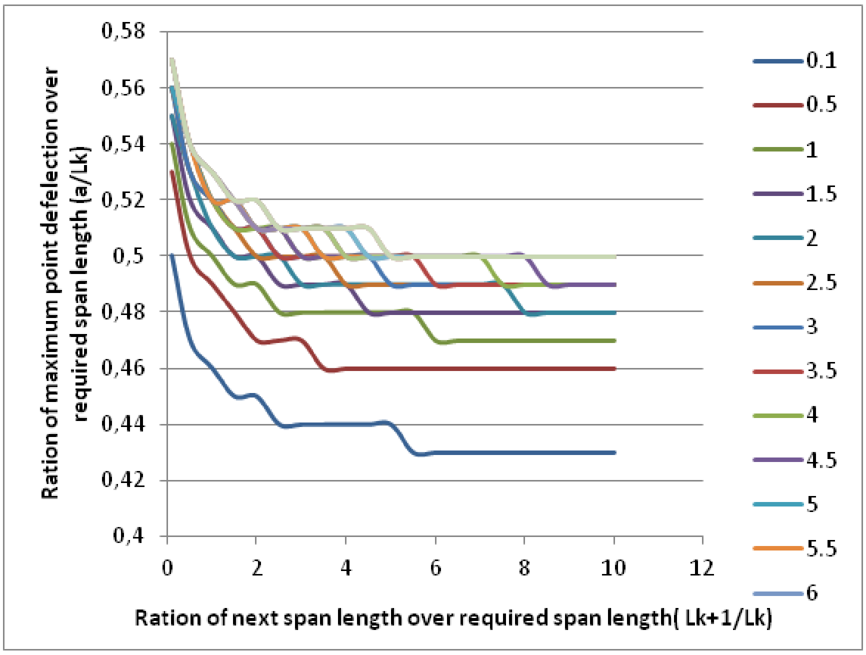

Figure 2. Location ration of maximum point of deflection ($\frac{a}{L_k}$) when required span is located between two spans

In case the required span is at first or last, the same methodology was applied but taking one span ration as shown in figure 3:

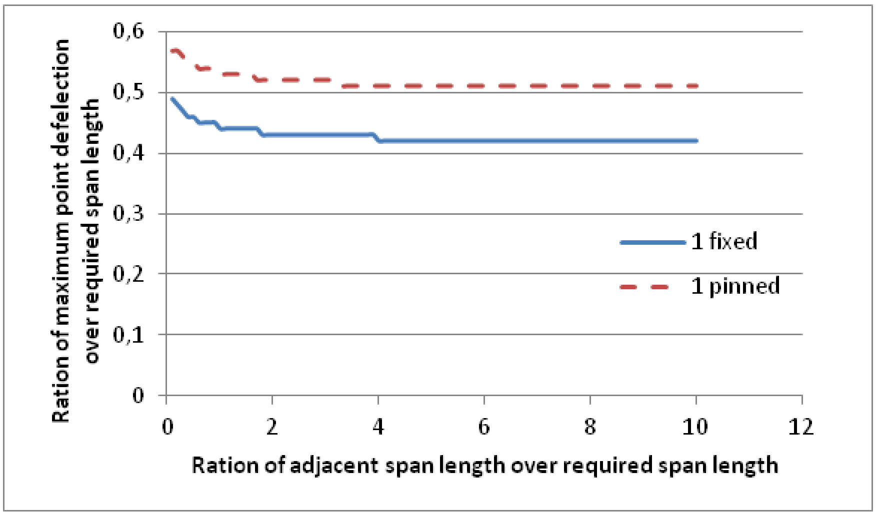

Figure 3. Location ration of maximum point of deflection ($\frac{a}{L_k}$) when required span is at start or at end

When location of maximum defection point is defined, the maximum deflection is easily found by using equation 24 or 25 and removing $C_{(X)L}$, $C_{(X)R}$ respectively since point location is under the moving load as follows:

$$\Delta_{MAX}=\frac{[2\ M_{Xk}\ L_k+M_{Lk}\ X_{MDP}+M_{Rk}\ (L_k-X_{MDP})]\ X_{MDP}\ (L_k-X)}{6\ EI\ L} \quad (26)$$

X_{MDP}: Distance from left support to Maximum deflection point.

Δ_{MAX} : Maximum deflection

8 ILLUSTRATION EXAMPLE AND RESULTS COMPARISON WITH STAAD SOFTWARE

Three spans bridge girder example is taken to demonstrate the accuracy of previous equations, STAAD software is used to justify the results, and each span is 20m length, rectangle cross section (1m ×0.5m), H20 truck is used as shown in below figure 4:

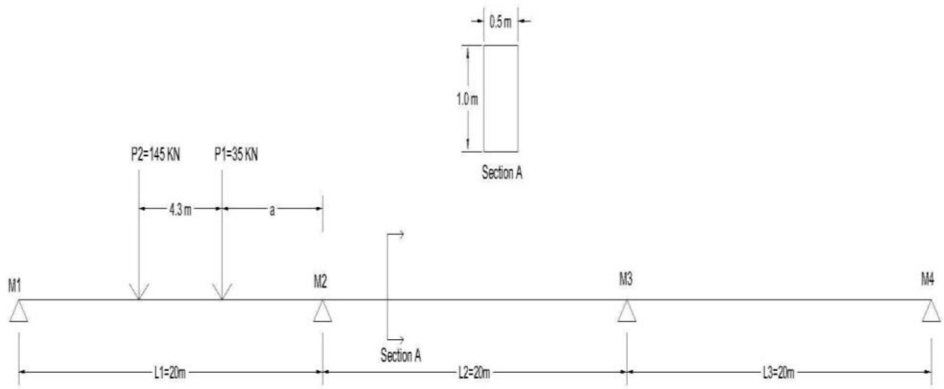

Figure 4. Three span bridge girder example illustration

At STAAD model, short interval is taken (0.10 m) to demonstrate the accuracy of used equations.

[R] =

1	0	0	0
20	80	20	0
0	20	80	20
0	0	0	1

$[R]^{-1}$ =

1	0	0	0
-0.267	0.01333	-0.0033	0.0667
0.0667	-0.0033	0.01333	-0.267
0	0	0	1

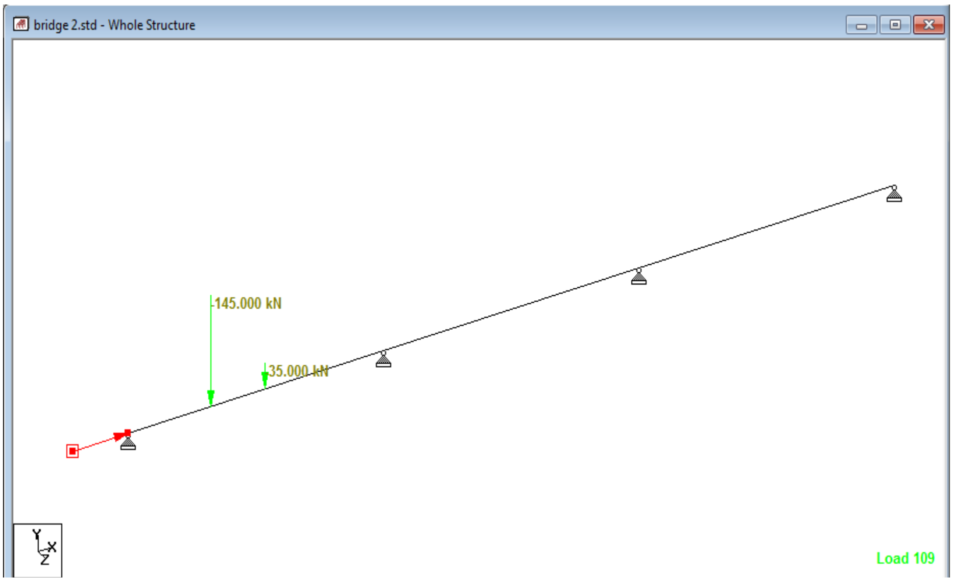

Figure 5. Moving load illustration, STAAD

8.1 Maximum positive moment at support 2 (M2) and support 3 (m3) when loadings are at span 1 and span 3 respectively

$$A=\sum_{i=1}^{2}\frac{P_i\times[3\times\left(R^{-1}_{(2,2)}-0\text{ (there is no moment at first support, pinned)}\right)]}{L_1}$$

$$=\frac{35\times[3\times(0.013333-0)]}{20}+\frac{145\times[3\times(0.013333-0)]}{20}=0.36$$

$$B=\sum_{i}^{2}\frac{P_i\left[6\,x_{i-1}\left(R^{-1}_{(2,2)}-0\text{ (there is no moment at first support}\right)-6\,L_1\,R^{-1}_{(2,2)}\right]}{L_1}$$

$$=\frac{35\times[6\times0\times(0.013333-0)-6\times20\times0.013333]}{20}+$$

$$\frac{145\times[6\times4.3\times(0.013333-0)-6\times20\times0.013333]}{20}=-2.8-9.1058=-11.9058$$

$$C=\sum_{i=1}^{2}\frac{P_i\times\left[0\times\left(L_1^2-3\times x_{i-1}^2\right)+R^{-1}_{(2,2)}\times\left(3\times x_{i-1}^2+2\times L_1{}^2-6\times x_{i-1}\times L_1\right)\right]}{L_1}$$

$$=\frac{35\times\left[0+0.013333\times\left(3\times0+2\times20^2-6\times0\times20\right)\right]}{20}$$

$$+\frac{145\times\left[0+0.013333\times\left(3\times4.3^2+2\times20^2-6\times4.3\times20\right)\right]}{20}=51.48$$

$$a_{max}=\frac{-B-\sqrt{B^2-4\times A\times C}}{2\times A}=\frac{11.9058-\sqrt{-11.9058^2-4\times0.36\times51.48}}{2\times0.36}=5.115\ m$$

M_2=0(there is no moment at first support, pinned)×

$$\sum_{i=1}^{2}\frac{P_i\times\left[L_1^2\times(a_{max}+x_{i-1})-(a_{max}+x_{i-1})^3\right]}{L_1}+R^{-1}_{(k+1,k+1)}\times$$

$$\sum_{i=1}^{2}\frac{P_i\times\left[(a_{max}+x_{i-1})^3+2\times(a_{max}+x_{i-1})\times L_1^2-3\times(a_{max}+x_{i-1})^2\times L_1^2\right]}{L_1}=$$

$$0+\frac{35\times\left[(5.115+0)^3+2\times(5.115+0)\times20^2-3\times(5.115+0)^2\times20\right]}{20}+$$

$$\frac{145\times\left[(5.115+4.3)^3+2\times(5.115+4.3)\times20^2-3\times(5.115+4.3)^2\times20\right]}{20}$$

$$=0.013333\times(4648.055+22098.24)=356.53\ KN.m$$

Since first and last support are pinned and all span lengths are the same, then the maximum support moment M2 occurs when loading at first span, no need to check M2 when loading are in span 2, M3 is same as M2 due to symmetry.

As shown in Figure 6, maximum positive moment at support 2 is 355.85 KN.m which is too close with calculated value 356.53 KN.m.

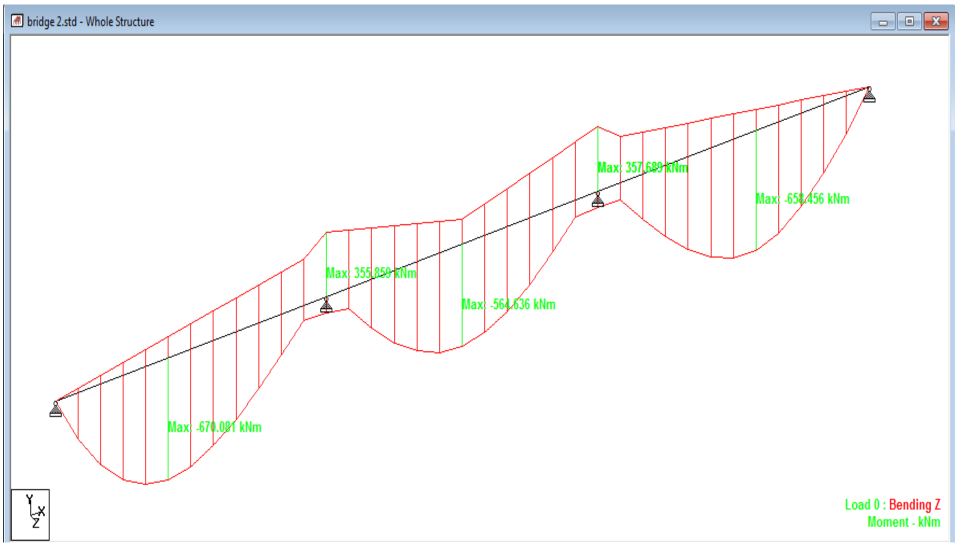

Figure 6. Moment envelope diagram, STAAD

8.2 Positive shear envelope equation

Positive shear envelope equation is the maximum of two equations as follows:
Positive shear envelope equation at span 1 = MAX $(V(x)_{p1}, V(x)_{p2})$

$$V1(x)_{p1}=\frac{\sum_i^2 P_i\times(a+x_{i-1})+M_{L1}-M_{R1}}{L_1}-P_2=\frac{35\times(a)+145\times(a+4.3)+0-M2}{L_1}-145$$

$$V1(x)_{p2}=\frac{\sum_i^2 P_i\times(a+x_{i-1})+M_{L1}-M_{R1}}{L_1}=\frac{35\times(a)+145\times(a+4.3)+0-M2}{L_1}$$

Where M2 =

$$0(\text{there is no moment at first support, pinned})\times\sum_{i=1}^{2}\frac{P_i\times\left[L_1^2\times\left(a+x_{i-1}\right)-\left(a+x_{i-1}\right)^3\right]}{L_1}+$$

$$R_{(2,2)}^{-1}\times\sum_{i=1}^{2}\frac{P_i\times\left[\left(a+x_{i-1}\right)^3+2\times\left(a+x_{i-1}\right)\times L_1^2-3\times\left(a+x_{i-1}\right)^2\times L_1\right]}{L_1}=0.01333\times\{$$

$$\frac{35\times\left[(a)^3+2\times(a)\times 20^2-3\times(a)^2\times 20\right]}{20}+$$

$$\frac{145\times\left[(a+4.3)^3+2\times(a+4.3)\times 20^2-3\times(a+4.3)^2\times 20\right]}{20}\}$$

At same manner, positive shear envelope equation for span 2 is found as follows:

$$V2(x)_{p1}=\frac{\sum_i^2 P_i\times(a+x_{i-1})+M_{L1}-M_{R1}}{L_2}-P_2=\frac{35\times(a)+145\times(a+4.3)+M2-M3}{L_2}-145$$

$$V2(x)_{p2}=\frac{\sum_i^2 P_i\times(a+x_{i-1})+M_{L1}-M_{R1}}{L_2}=\frac{35\times(a)+145\times(a+4.3)+M2-M3}{L_2}$$

Where

$$M2=R^{-1}_{(2,2)}\times\sum_{i=1}^{2}\frac{P_i\times\left[L_2^2\times\left(a+x_{i-1}\right)-\left(a+x_{i-1}\right)^3\right]}{L_2}+R^{-1}_{(2,3)}\times$$

$$\sum_{i=1}^{2}\frac{P_i\times\left[(a+x_{i-1})^3+2\times(a+x_{i-1})\times L_2^2-3\times(a+x_{i-1})^2\times L_2\right]}{L_2}$$

$$=0.013333\times\left\{\frac{35\times\left[20^2\times(a)-(a)^3\right]}{20}+\frac{145\times\left[20^2\times(a+4.3)-(a+4.3)^3\right]}{20}\right\}$$

$$-0.00333\times(\frac{35\times\left[(a)^3+2\times(a)\times20^2-3\times(a)^2\times20\right]}{20}$$

$$\frac{145\times\left[(a+4.3)^3+2\times(a+4.3)\times20^2-3\times(a+4.3)^2\times20\right]}{20})$$

$$M3=R^{-1}_{(3,2)}\times\sum_{i=1}^{2}\frac{P_i\times\left[L_2^2\times\left(a+x_{i-1}\right)-\left(a+x_{i-1}\right)^3\right]}{L_2}+$$

$$R^{-1}_{(3,3)}\times\sum_{i=1}^{2}\frac{P_i\times\left[(a+x_{i-1})^3+2\times(a+x_{i-1})\times L_2^2-3\times(a+x_{i-1})^2\times L_2\right]}{L_2}$$

$$=-0.00333\times\left\{\frac{35\times\left[20^2\times(a)-(a)^3\right]}{20}+\frac{145\times\left[20^2\times(a+4.3)-(a+4.3)^3\right]}{20}\right\}$$

$$+0.013333\times\{\frac{35\times\left[(a)^3+2\times(a)\times20^2-3\times(a)^2\times20\right]}{20}+$$

$$\frac{145\times\left[(a+4.3)^3+2\times(a+4.3)\times20^2-3\times(a+4.3)^2\times20\right]}{20}\}$$

To plot Shear envlope equation, excel sheet is used where a is subistuited with L-X where X starts from zero to end of span length (20 m) (from left to right).

8.3 Negative shear envelope equation

$$V(x)_N = P_{Lk} - \sum_{i}^{c} P_i + \sum_{i=1}^{n} P_{i-1} \qquad (26)$$

Negative shear envelope equation is the maximum of two equations like positive shear envelope as follows:
Negative shear envelope equation at span 1 = MAX $(V(x)_{N1}, V(x)_{N2})$

$$V1(x)_{N1} = \frac{\sum_i^2 P_i \times (a+x_{i-1}) + M_{L1} - M_{R1}}{L_1} - P_2 - P1 = \frac{35\times(a)+145\times(a+4.3)+0-M2}{L_1} - 145 - 35$$

$$V1(x)_{N2} = \frac{\sum_i^2 P_i \times (a+x_{i-1}) + M_{L1} - M_{R1}}{L_1} - P1 = \frac{35\times(a)+145\times(a+4.3)+0-M2}{L_1} - 35$$

$$V2(x)_{N1} = \frac{\sum_i^2 P_i \times (a+x_{i-1}) + M_{L1} - M_{R1}}{L_2} - P_2 - P1 = \frac{35\times(a)+145\times(a+4.3)+M2-M3}{L_2} - 145 - 35$$

$$V2(x)_{N2} = \frac{\sum_i^2 P_i \times (a+x_{i-1}) + M_{L1} - M_{R1}}{L_2} - P1 = \frac{35\times(a)+145\times(a+4.3)+M2-M3}{L_2} - 35$$

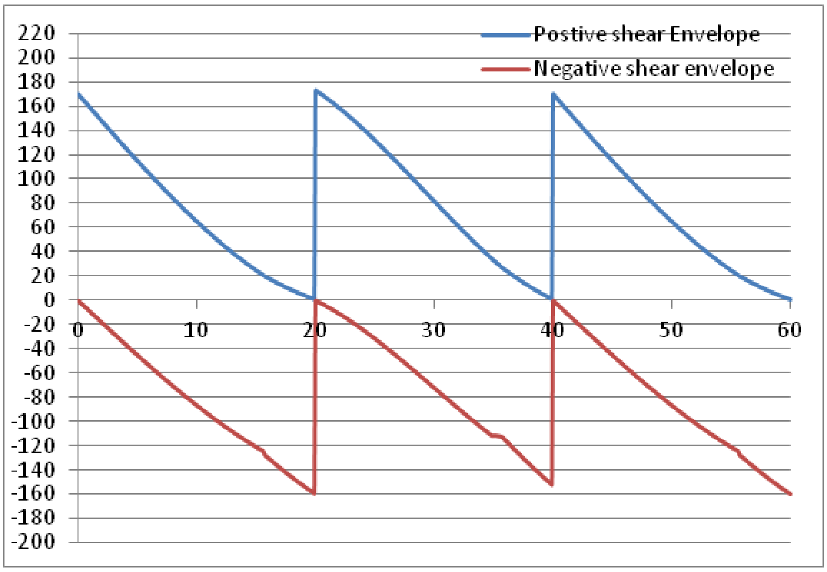

Figure 7. Positive and negative shear envelope diagram using equations

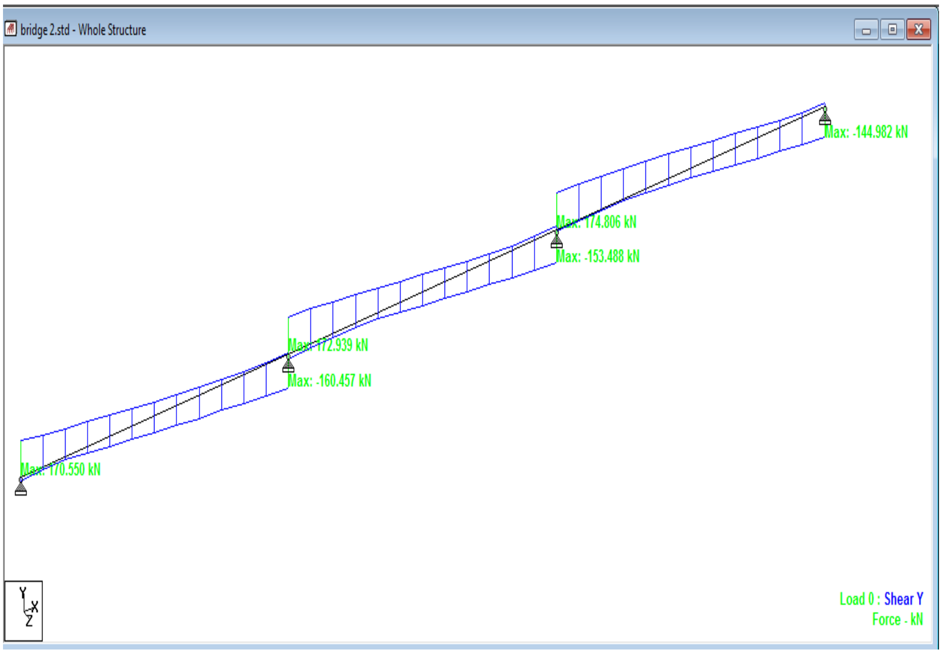

Figure 8. Shear envelope diagram, STAAD

As shown in above figures (7&8), the shear envelope calculated by presented equations in this paper and STAAD Software are almost the same.

8.4 Negative moment envelope equation

$$M(x)_N = M_{Lk} - P_{Lk} \times x + \sum_{i=c}^{n+1} P_i \times x_{i-1} \tag{27}$$

Negative Moment envelope equation is the maximum of two equations as follows:

Negative moment envelope equation at span 1 = MAX $(M1(x)_{N1}, M1(x)_{N2})$
Negative moment envelope equation at span 2 = MAX $(M2(x)_{N1}, M2(x)_{N2})$

$$M1(x)_{N1} = M_1 - \frac{P_1 \times a + P_2 \times (a+4.3) + M_1 - M_2}{L_1} \times (L_1 - a) + P_1 \times 4.3$$

$$= 0 - \frac{35 \times a + 145 \times (a+4.3) + 0 - M_2}{20} \times (20 - a) + P_1 \times 4.3$$

$$M1(x)_{N2} = M_1 - \frac{P_1 \times a + P_2 \times (a+4.3) + M_1 - M_2}{L_1} \times (L_1 - a - 4.3)$$

$$= 0 - \frac{35\times a+145\times(a+4.3)+0-M_2}{20}\times(20-a-4.3)$$

$$M2(x)_{N1}=M_2-\frac{P_1\times a+P_2\times(a+4.3)+M_2-M_3}{L_2}\times(L_2-a)+P_1\times 4.3$$

$$= M_2-\frac{35\times a+145\times(a+4.3)+M2-M_3}{20}\times(20-a)+P_1\times 4.3$$

$$M2(x)_{N2}=M_2-\frac{P_1\times a+P_2\times(a+4.3)+M_2-M_3}{L_2}\times(L_2-a-4.3)$$

$$= M_2-\frac{35\times a+145\times(a+4.3)+M_2-M_3}{20}\times(20-a-4.3)$$

Using Excel sheet, above equations were plotted in two diagrams, first one is the Negative moment envelope diagram at span 1, the second is for span 2 as follows:

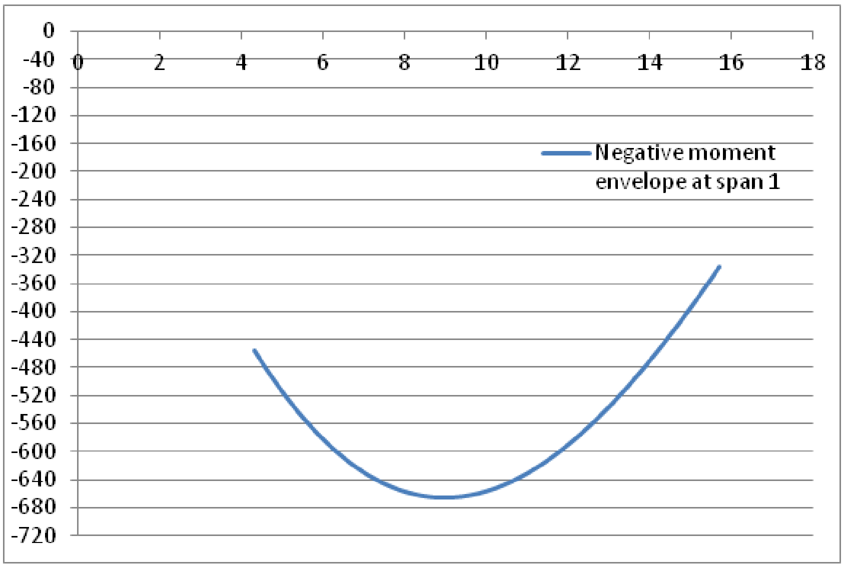

Figure 9. Negative moment envelope diagram at span 1

Negative envelope moment diagram at span 3 is similar at span 1 due to symmetry, Maximum Negative moment values shown in above two figures (9 &10) are too close with those calculated by STAAD model shown in figure 6.

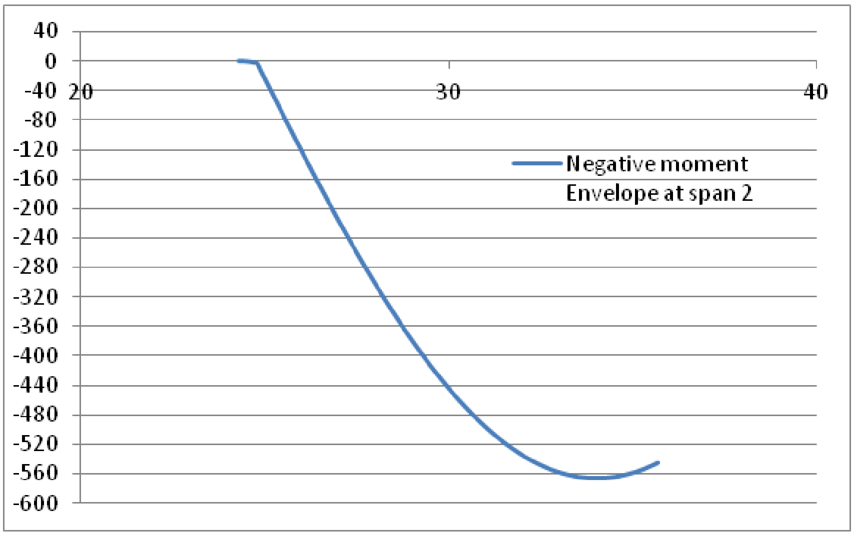

Figure 10. Negative moment envelope diagram at span 2

8.5 Envelope deflection equation and maximum deflection point

Maximum deflection points at span 1 and span 3 occurs at distance of 0.53 of span length as illustrated in figure 11:

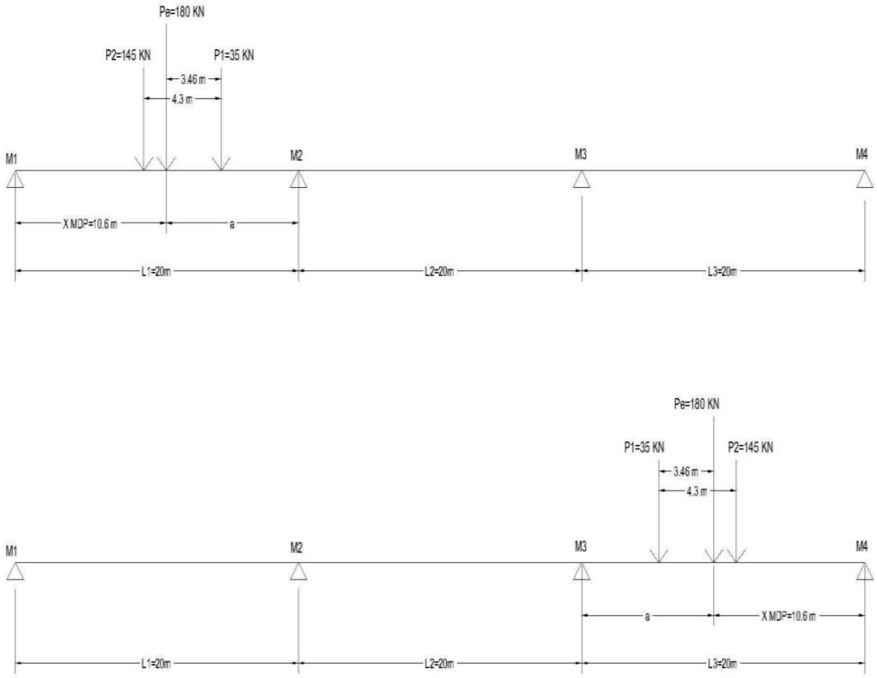

Figure 11. Load arrangement at span 1 and span 3 for maximum point of deflection

Equivalent moving load Pe (180 KN) was calculated by taking equilibrium moment equations around P1 and P2.

$$\Delta_{MAX}=\frac{[2\times M_{Xk}\times L_k+M_{Lk}\times X_{MDP}+M_{Rk}\times(L_k-X_{MDP})]\times X_{MDP}\times(L_k-X_{MDP})}{6\times EI\times L} \quad (29)$$

$$\Delta_{MAX}=\frac{[2\times M_{X1}\times L_1+M_1\times X_{MDP}+M_2\times(L_1-X_{MDP})]\times X_{MDP}\times(L_1-X_{MDP})}{6\times EI\times L_1}$$

$$=\frac{[2\times M_{X1}\times 20+0\times 10.6+M_2\times(20-10.6)]\times 10.6\times(20-10.6)}{6\times EI\times 20}$$

Where $M_2=R^{-1}_{(2,1)}\times\frac{P_e\times[L_1{}^2\times a-a^3]}{L_1}+R^{-1}_{(2,2)}\times\frac{P_e\times[a^3+2\times a\times L_1{}^2-3\times a^2\times L_1]}{L_1}=$

$$0\times\frac{P_e\times[L_1{}^2\times a-a^3]}{L_1}+0.013333\times\frac{180\times[9.4^3+2\times 9.4\times 20^2-3\times 9.4^2\times 20]}{20}$$

$= 365.86$ KN.m

M_{X1} is the negative moment under Pe, using simple equilibrium equation, it is equal to -702.886 KN.m

$\Delta_{MAX}=20489.6\ /\ EI$

$I=\frac{0.50\times 1^3}{12}=0.041667$ m^4

E= 21718500 KPa as defined by STAAD model

$\Delta_{MAX}=20489.6\ /\ 904944 = -22.64$ mm.

Maximum Deflection in span 1 according to STAAD model is -22.279 mm, the two values are almost the same

Maximum Deflection at span 1 is the same at span 3 due to symmetry, maximum deflection at span 2 (for geometry and load arrangement refer to figure 12) is calculated at same manner but the ration of maximum deflection point will be 0.50 as follows:

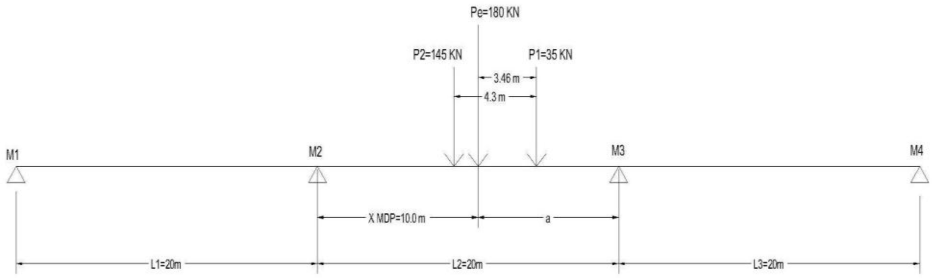

Figure 12. Load arrangement at span 2 for maximum point of deflection

$$\Delta_{MAX}=\frac{[2\times M_{X2}\times L_2+M_2\times X_{MDP}+M_3\times(L_2-X_{MDP})]\times X_{MDP}\times(L_2-X_{MDP})}{6\times EI\times L_2}=$$

$$\frac{[2\times M_{X2}\times 20+M_2\times 10+M_3\times(20-10)]\times 10\times(20-10)}{6\times EI\times 20}$$

Where

$$M_2=R^{-1}_{(2,2)}\times\frac{P_e\times[L_1{}^2\times a-a^3]}{L_1}+R^{-1}_{(2,3)}\times\frac{P_e\times[a^3+2\times a\times L_1{}^2-3\times a^2\times L_1]}{L_1}=0.013333\times$$

$$\frac{180\times[20^2\times 10-10^3]}{L_1}+-0.00333\times\frac{180\times[10^3+2\times 10\times 20^2-3\times 10^2\times 20]}{20}$$

$= 270.081$ KN.m

$M_2 = M_3$ due to summetry.

M_{X2} is the negative moment under Pe, using simple equilibrium equation, it is equal to -629.919 KN.m

Δ_{MAX}=16495.95 / EI =16495.95 / 904944 = 18.22 mm.

Maximum Deflection in span 2 according to STAAD model is -17.384 mm, the two values are too close.

9 CONCLUSION

As shown in above calculations, these derived equations prove its accuracy particularly when compared with STAAD software, so this paper gives general reference for all bridge and structural engineers for moving load analysis without any restrictions or limitations, making bridge girder design more feasible and practical without using any software.

Also bridge software programmers can use these direct equations in their software due its accuracy and writing code simplicity.

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