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LAMINATION SCHEME AND BOUNDARY CONDITIONS EFFECTS ON THE FREE VIBRATION OF LAMINATED COMPOSITE BEAMS

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ABSTRACT: In this study, the effects of both lamination scheme and boundary conditions on the natural frequencies of free vibration of laminated composite beams were investigated. The problem is analyzed and solved using the energy approach which is formulated by a finite element model. Lamination schemes for symmetric and non-symmetric laminated beams were studied. Six boundary conditions are considered; clamped _free (CF), hinged _hinged (HH), clamped _clamped (CC), hinged _clamped (HC), hinged _free (HF), free _free (FF). Each beam has either movable ends or immovable ends. It is found that both symmetrically and anti-symmetrically laminated beams of similar size and end conditions have equal natural frequencies which, generally, decrease as the angle of orientation increases. Also, It is found that the more constrained beams have the higher values of natural frequencies of transverse vibration. However, the free-free and hinged-free beams are found to have the highest frequencies of transverse vibration amongst all beams although they look less constrained. This behavior is due to the fact that the first mode of the two beams is equal zero (rigid body motion), and replaced by the second mode to be the fundamental mode. The values of the natural frequencies of longitudinal modes are found to be the same for all beams with movable ends since they are generated by longitudinal movements only. But for immovable ends, the clamped-free and hinged-free beams have equal frequencies in longitudinal vibration, and those of the other beams are also the same.

KEYWORDS: Finite Element Method, First Order Shear Deformation Theory, Free Vibration, Laminated Composites, Natural Frequencies.

1 INTRODUCTION

Laminar composites are those having alternating layers of material bonded together in some manner and include thin coatings, thicker protective surfaces, claddings, bimetallic, laminates, and sandwiches. Laminated composite beams are increasingly being used in many engineering applications in the fields of mechanical and civil engineering, transportation vehicles, marine, aviation and aerospace.

The papers, which are presented here as references, address the problem of the free vibration of laminated composite beams. A theoretical analysis of the vibration of composite beams with solid cross sections was also presented by Teoh and Huang [1], Chandrashekara et al. [2], Abramovich [3]. In those analyses, the equations of motion were based on a Timoshenko beam model (shear deformation considered). Numerical results showed the effect of the shear deformation and fiber orientation on the natural frequencies. Again, Abramovich and Livshits [4] presented exact solutions for the free vibration of non-symmetrically laminated cross-ply composite beams. Marur and Kant [5] and [6], and McCarthy et al [7] applied higher order shear deformation theories to solve the problem of the free vibration of composite beams.

The first-order shear deformation theory was used by Teboub and Hajela [8] to analyze the free vibration of generally layered composite beams. Hodges et al. [9] presented two different methods, which were simple analytical method and finite element method for the prediction of the natural frequencies and mode shapes of composite beams. In addition to the references mentioned above, references [10, 11, and 12] applied different techniques of the finite element method for the same problem.

2 MATHEMATICAL FORMULATIONS

The time-dependent axial and transverse displacements fields are:

$$\left.\begin{aligned} U(x,z,t) &= u(x,t) + z\phi(x,t) \\ W(x,z,t) &= w(x,t) \end{aligned}\right\} \tag{1}$$

Where, u and w are the axial and transverse displacements at the mid-plane, z is the perpendicular distance from the mid-plane to the layer plane, ϕ is the rotation of a plane after deformation, and t is the time. The strain- displacement relations are:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \\ \varepsilon_5 &= \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = \frac{\partial w}{\partial x} + \phi \end{aligned}\right\} \tag{2}$$

Where the subscripts have the same meanings as those used in 3-D elasticity formulation, i.e. ε_1 is the axial or longitudinal strain, and ε_5 is the through-thickness shear strain. The stress-strain relationship of a lamina can be shown as:

$$\{\sigma_i\} = [\overline{C}_{ij}]\{\varepsilon_i\} \tag{3}$$

Where,

$$\left.\begin{aligned} \{\sigma\}^T &= \{\sigma_1 \quad \sigma_5\} \\ [\overline{C}_{ij}] &= \begin{bmatrix} \overline{C}_{11} & 0 \\ 0 & \overline{C}_{55} \end{bmatrix} \\ \{\varepsilon\}^t &= [\varepsilon_1 \quad \varepsilon_5] \end{aligned}\right\} \tag{4}$$

The elastic constants $[\overline{C}_{11}]$ and $[\overline{C}_{55}]$ for orthotropic beams can be expressed as:

$$\left.\begin{aligned} \overline{C}_{11} &= C_{11}.\cos^4\theta + C_{22}.\sin^4\theta + 2(C_{12} + 2.C_{66}).\sin^2\theta.\cos^2\theta \\ \overline{C}_{55} &= C_{44}.\sin^2\theta + C_{55}.\cos^2\theta \end{aligned}\right\} \tag{5}$$

Where;

$$C_{66} = G_{12};\ \mathrm{C}_{44} = G_{23}; \mathrm{C}_{55} = G_{13};\ \mathrm{C}_{12} = \nu_{12}\mathrm{C}_{22} = \nu_{21}\mathrm{C}_{11} \tag{6}$$

By applying the energy approach for the beam element shown in Fig. (2), the strain energy stored is given by:

$$U_s = \frac{1}{2}\int_e \{\varepsilon\}^T \{\sigma\} dV \tag{7}$$

Where, $dV = bdxdz$, and the subscript, e, means one element.
Also, the kinetic energy is found as follows;

$$K.E. = \frac{1}{2} b\rho \int_e \left[\frac{\partial^2 W}{\partial t^2} W + \frac{\partial^2 U}{\partial t^2} U \right] dx\, dz \tag{8}$$

The degrees of freedom at each node are; the axial displacement u, deflection w, and rotation ϕ , and can be written in terms of their nodal values as follows:

$$[u, w, \phi] = \sum_{i=1}^{i=3} [N_i u_i, N_i w_i, N_i \phi_i] \tag{9}$$

Where, N_i is shape function and assumed as a second-order polynomial for a three-noded element as:

$$N_i = a_i + b_i x + c_i x^2 \qquad (i = 1,2,3) \tag{10}$$

The constants a_i, b_i, and c_i can be computed for each element from the following data:

$$N_i = \begin{cases} 1 \\ 0 \end{cases} \quad \text{at} \quad \begin{cases} x = x_i \\ x \neq x_i \end{cases} \quad (i = 1,2,3) \tag{11}$$

Eqns. (7), and (8) leads to the final form of the non-dimensional element stiffness and inertia matrices [K]e ,and [M]e respectively. The individual element stiffness and inertia matrices [K]e and [M]e must be linked together or assembled to characterize the unified behavior of the entire beam. Therefore, The global stiffness and inertia matrices are given respectively by,

$$\left. \begin{aligned} [K] &= \sum_{n=1}^{N} [K]^e \\ [M] &= \sum_{n=1}^{N} [M]^e \end{aligned} \right\} \tag{12}$$

Where, N is the total number of beam elements.

The solution can be obtained after the incorporation of boundary conditions which will modify both stiffness and inertia matrices. Thus, the non-dimensionalized natural frequencies can be determined from the relation:

$$\left| [M]^{-1} [K] - \omega^2 I \right| = 0 \tag{13}$$

Where, I is an identity matrix, and ω is the non-dimensional natural frequencies, which can be computed by computing the square root of the eigenvalues of the matrix $[M]^{-1}[K]$ using a suitable computer program (Here MATLAB was used).

3 MATERIAL

AS/3501-6 graphite-epoxy material was used for all numerical results because of its wide applications in modern industries. The mechanical properties of this material are tabulated in Table (1).

Table 1. Mechanical Properties of AS/3501-6 graphite-epoxy material

Property	**Magnitude**
E_1	145 GN/m^2
E_2	9.6 GN/m^2
G_{12}	4.1 GN/m^2
G_{13}	4.1 GN/m^2
G_{23}	3.4 GN/m^2
Poisson's ratio (ν)	0.3
Density (ρ)	1520 kg/m^3

4 METHOD VALIDITY

In order to check the validity of the present method, some comparisons were performed. Table (2) and Table (3) show comparisons with the results of some past works. For the hinged-hinged beam, a percentage difference of less than *0.16%* was recorded for the fundamental frequency, and less than *0.54%* for both fixed-free and fixed-fixed beams. This difference was observed to increase as the mode order increases (less than *1.4%*) for the seventh mode for all beams considered.

Table 2. Non-dimensional fundamental frequencies [$\bar{\omega} = \omega.L^2\sqrt{\rho/E_1h^2}$] of symmetric [$\theta/-\theta/-\theta/\theta$] angle-ply beams (L/h = 15)

Beam Type \ θ (deg.)		0	15	30	45	60	75	90
HH	present	2.654	2.509	2.102	1.535	1.010	0.759	0.730
	Ref. [2]	2.656	2.510	2.103	1.536	1.012	0.761	0.732
CC	Present	4.839	4.655	4.092	3.182	2.1996	1.683	1.622
	Ref. [2]	4.848	4.663	4.098	3.184	2.1984	1.681	1.620
HF	present	4.090	3.870	3.251	2.382	1.5716	1.181	1.136
	Ref. [2]	4.093	3.872	3.253	2.384	1.5738	1.184	1.138
FF	Present	5.889	5.574	4.687	3.438	2.2702	1.707	1.642
	Ref. [2]	5.892	5.577	4.689	3.440	2.2730	1.710	1.645
CF	present	0.981	0.924	0.767	0.554	0.3625	0.271	0.261
	Ref. [2]	0.982	0.924	0.767	0.555	0.3631	0.272	0.261
CH	Present	3.725	3.555	3.054	2.301	1.5502	1.174	1.130
	Ref. [2]	3.730	3.559	3.057	2.303	1.5511	1.175	1.131

Table 3. Non-dimensional frequencies [$\bar{\omega} = \omega.L^2\sqrt{\rho/E_1h^2}$] of symmetric [0/ 90/ 90/ 0] cross-ply beams (L/h = 10)

Mode No.	Hinged-hinged (Immovable)		Fixed-free (Immovable)		Fixed-fixed (Immovable)	
	Present	Ref. [4]	Present	Ref. [4]	Present	Ref. [4]
1	2.3157	2.3194	0.8866	0.8819	3.6855	3.7576
2	6.9813	7.0029	4.1062	4.0259	7.7244	7.8718
3	12.004	12.037	8.9536	9.1085	12.381	12.573
4	17.010	17.015	11.504*	12.193*	17.192	17.373
5	22.015	21.907	13.924	14.080	22.119	22.200
6	23.007*	23.007*	18.980	18.980	23.007*	23.007*

Mode No.	Hinged-hinged (Immovable)		Fixed-free (Immovable)		Fixed-fixed (Immovable)	
	Present	Ref. [4]	Present	Ref. [4]	Present	Ref. [4]
7	27.094	27.094	24.037	24.037	27.125	27.125

(*) *Modes with predominance of longitudinal vibration*

5 NUMERICAL RESULTS

5.1 Lamination Scheme:

Table 4. The first three non-dimensional modes of free vibration of symmetric $[\theta/-\theta/-\theta/\theta]$ laminated beams with immovable ends. L/h=10

Angle (θ)	Mode No.	Beam type					
		CF	HH	CC	HC	HF	FF
30^o	1 2 3	0.7465 3.7279 8.4193	1.9918 6.4128 11.4744	3.4380 7.4386 12.7816	2.7113 6.9645 11.7816	3.0503 7.9206 13.1707	4.3728 9.5329 14.9573
60^o	1 2 3	0.3596 2.0893 5.3017	0.9943 3.6853 7.4807	2.0601 5.0523 8.8047	1.4899 4.3780 8.1620	1.5370 4.5879 8.6142	2.2088 5.5663 9.8029
90^o	1 2 3	0.2597 1.5522 4.0677	0.7224 2.7525 5.7774	1.5544 3.9654 7.1377	1.1020 3.3519 6.4646	1.1184 3.4324 6.6580	1.6077 4.1694 7.5785

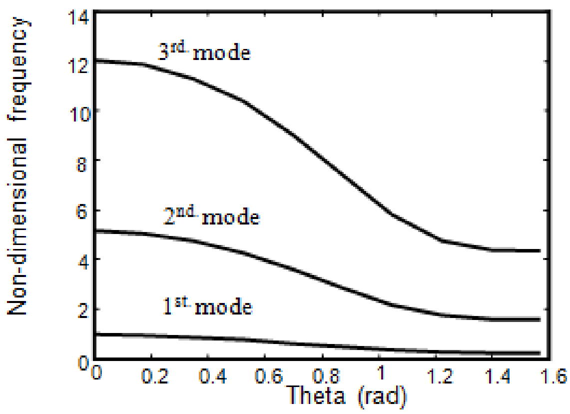

Figure 1. Clamped-free beam

Table (4) shows that the values of non-dimensional natural frequencies of various beams generally decrease as the angle of orientation of fibers with respect to the longitudinal axis of the beam are increased.

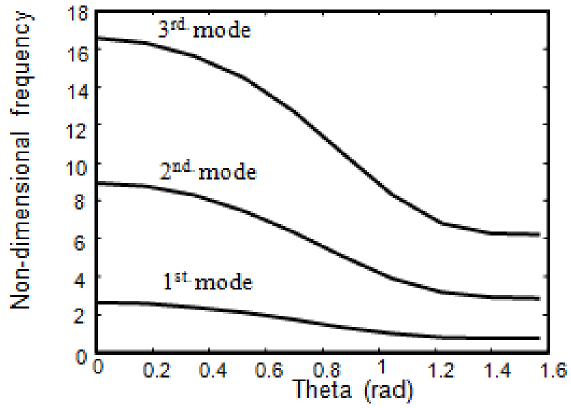

Figure 2. Hinged-hinged beam

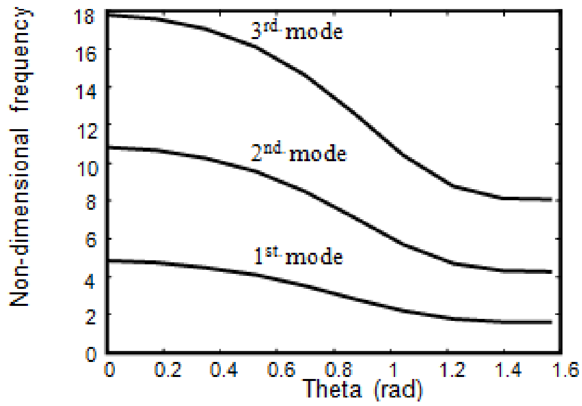

Figure 3. Clamped-clamped beam

Similar values of frequencies for symmetric $[\theta/-\theta/-\theta/\theta]$ laminated beams with immovable ends and aspect ratio of 10 are plotted against the angle of orientation for the range from 0 up to 90 degrees in Figure 1 to Figure 6. The influence of fiber orientation becomes more noticeable as the mode order increases, and significant variations of frequencies were observed up to an angle of approximately 70 degrees. Beyond this angle, the variations in the frequencies are very small.

Increasing angle of orientation to more than 70 degrees leads to increase the coupling between bending and stretching effect, which causes the laminated beam to be stiffer, and thus the variation in natural frequencies decreases. In addition, the values of non-dimensional natural frequencies of the longitudinal modes of free vibration are observed to decrease as the angle of fibers orientation is increased.

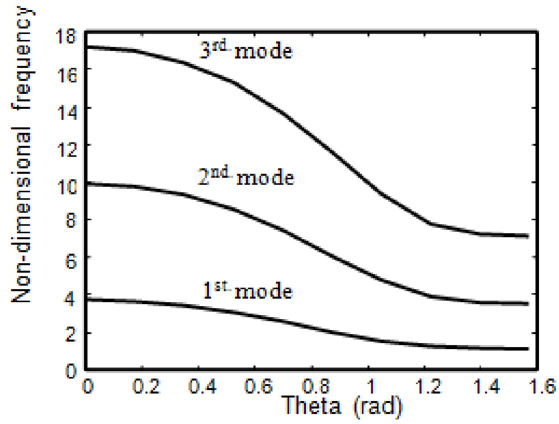

Figure 4. Hinged-clamped beam

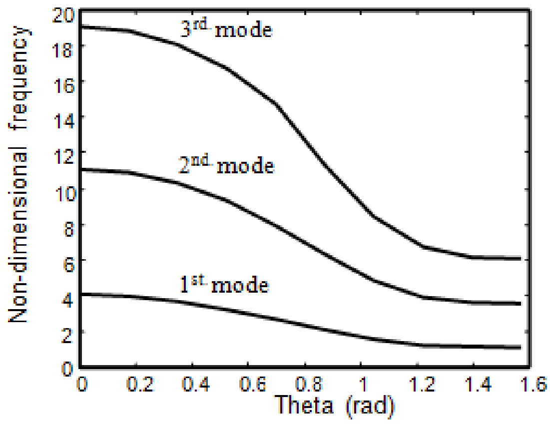

Figure 5. Hinged-free beam

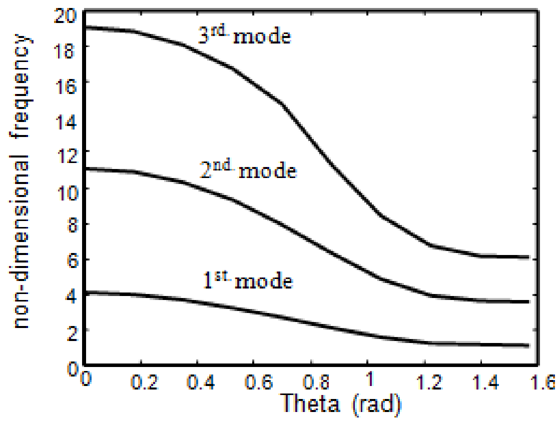

Figure 6. Hinged-free beam

5.2 Boundary conditions

Table (5) shows the first ten modes of free vibration of cross-ply laminated composite beams with immovable ends (i.e. axial movement is restricted), while Table (6) shows those for movable ends.

Generally, it is found that more constrained beams have high values of natural frequencies. However, the free-free and hinged-free beams are found to have the highest frequencies amongst all beams although they look less constrained. This behavior is due the fact that the first mode of the two beams is equal zero and replaced by the second mode. The fundamental mode shapes of both beams are straight lines, Figure 7, and this due to the rigid motion in this mode where there is no vibrating motion.

Table 5. Non-dimensional natural frequencies $\left[\bar{\omega} = \omega\sqrt{\rho L^4 / E_1 h^2}\right]$ of a symmetric cross-ply [*90/-90/-90/90*] laminated beam with immovable ends, (*L/h = 10*)

Mode No.	BEAM TYPE					
	CF	HH	CC	HC	HF	FF
1	0.2597	0.7224	1.5544	1.1020	1.1184	1.6077
2	1.5522	2.7525	3.9654	3.3519	3.4324	4.1694
3	4.0539*	5.7774	7.1377	6.4646	4.0539*	7.5785
4	4.0677	8.1077*	8.1077*	8.1077*	6.6580	8.1077*
5	7.3503	9.4753	10.8006	10.1515	10.4868	11.5233
6	11.1503	13.5995	14.7896	14.2085	12.1616*	15.7969
7	12.1616*	16.2155*	16.2155*	16.2155*	14.6913	16.2155*
8	15.2783	17.9847	18.9972	18.5027	19.1205	20.2606
9	19.6134	22.5259	23.3543	22.9488	20.2694*	24.3234*
10	20.2694*	24.3234*	24.3234*	24.3234*	23.6776	24.8245

(*) *Modes with predominance of longitudinal vibration*

Table 6. Non-dimensional natural frequencies $\left[\bar{\omega} = \omega\sqrt{\rho L^4 / E_1 h^2}\right]$ of a symmetric cross-ply [90/-90/-90/90] laminated beam with movable ends, (L/h = 10)

Mode No.	CF	HH	CC	HC	HF	FF
1	0.2597	0.7224	1.5544	1.1020	1.1184	1.6077
2	1.5522	2.7525	3.9654	3.3519	3.4324	4.1694
3	4.0677	5.7774	7.1377	6.4646	6.6580	7.5785
4	7.3503	8.1077*	8.1077*	8.1077*	8.1077*	8.1077*
5	8.1077*	9.4753	10.8006	10.1515	10.4868	11.5233
6	11.1503	13.5995	14.7896	14.2085	14.6913	15.7969

Mode No.	CF	HH	CC	HC	HF	FF
7	15.2783	16.2155*	16.2155*	16.2155*	16.2155*	16.2155*
8	16.2155*	17.9847	18.9972	18.5027	19.1205	20.2606
9	19.6134	22.5259	23.3543	22.9488	23.6776	24.3234*
10	24.0758	24.3234*	24.3234*	24.3234*	24.3234*	24.8245

(*) *Modes with predominance of longitudinal vibration*

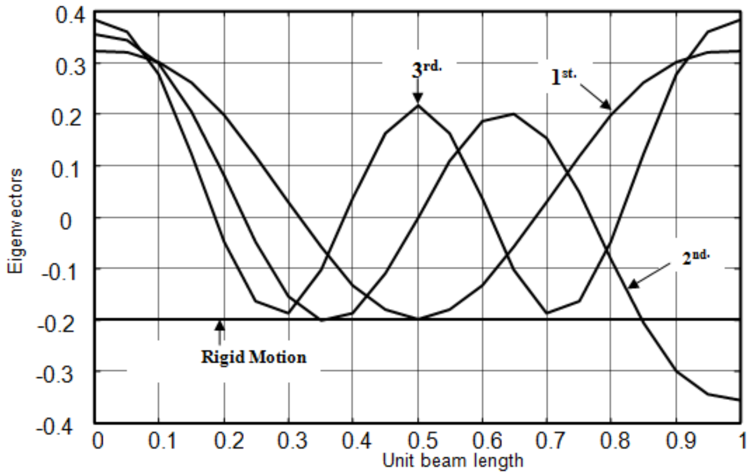

Figure 7. Transverse mode shapes of a symmetric of a free-free beam, L/h=10

6 CONCLUSIONS

The main conclusions are:

1. Similar beams, which are either symmetrically laminated $[\theta/-\theta/-\theta/\theta]$ or anti-symmetrically laminated $[\theta/-\theta/\theta/-\theta]$, have equal natural frequencies, since the coefficients $\overline{C}_{11}$ and $\overline{C}_{55}$ are equal for both cases (see Eqn. (5)).
2. The natural frequencies of a laminated beam generally decrease as the fiber orientation angle increases.
3. Increasing angle of orientation to more than *70* degrees leads to increase the coupling between bending and stretching effect, which causes the laminated beam to be stiffer, and thus the variation in natural frequencies decreases.
4. The values of natural frequencies of the longitudinal modes of free vibration are observed to decrease as the angle of fibers orientation is increased.
5. More restrained beams have high values of natural frequencies.
6. The free-free and hinged-free beams are found to have the highest frequencies amongst all beams although they look less constrained. This behavior is due to the fact that the first mode of the two beams is equal zero

(rigid body motion), and replaced by the second mode to be the fundamental mode.

7. The transverse modes are not affected by the longitudinal movements of the ends since these modes are generated by lateral movements only.
8. The values of the natural frequencies of longitudinal modes are found to be the same for all beams with movable ends since they are generated by longitudinal movements only.
9. Natural frequencies of the longitudinal vibration for the (CF) and (HF) beams are equal, and those of the other beams are also the same. This phenomenon occurs since both (CF) and (HF) beams with immovable ends are the same when restricted from executing longitudinal motion at the ends. Similarly, the rest of beams with immovable ends have the same longitudinal end conditions.

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