International Journal of Bridge Engineering (IJBE), Vol. 6, No. 3, (2018), pp. 45-60

STEEL BRIDGES WITH NEGATIVE SAG UNDER CONCENTRATED OR DISTRIBUTED MOVING MASS-LOADS

Theodore G. Konstantakopoulos[1] and George T. Michaltsos[2]

[1, 2] Department of Civil Engineering, National Technical University of Athens, Greece
e-mail: michalts@central.ntua.gr, theokons@teemail.gr

ABSTRACT: This work deals with the linear dynamic response of simply supported light (steel) bridges with negative sag under moving concentrated or distributed mass-loads of constant magnitude and velocity. The present analysis focuses on the determination of the influence limits of both the concentrated and the distributed mass-loads on the bridges' vibration in relation to their velocity and to the sag magnitude of the bridge. The individual and coupling effect of these parameters on the dynamic response of the bridge are thoroughly discussed herein. A variety of numerical examples allow one to draw important conclusions for structural design purposes.

KEYWORDS: Bridge dynamics; Moving mass-loads; Negative sag; Concentrated and distributed loads.

1 INTRODUCTION

The study of the influence of dynamic loads on elastic structures is a very old and complicated problem. The determination of the dynamic effect of moving loads on elastic structures and particularly on bridges is very complex. Many systems in civil engineering design, and especially bridges can be idealized as a flexible beam under a moving mass. The existence of a moving mass causes non-linearity and makes the problem particularly difficult.

A number of works have been reported during the last 100 years aiming to present reliable solutions for such a multi-parameter problem by using two different methods: the first one is to perform tests and the second is that of a pure theoretical investigation. In recent years, transport engineering has experienced serious advances characterized by increasingly higher speeds and weights of vehicles, which result to developed vibrations and dynamic stresses much larger than ever before. From a historical viewpoint, the problem of moving loads was first approximated for the case where the mass of the girder is negligible compared to the mass of a moving single load with constant magnitude [1-3].

Another extreme case where the mass of the moving load is negligible compared to the mass of the girder was originally studied by Krylov [4] and later by Timoshenko [5] and Lowan [6]. A more complicated problem including both the above parameters, i.e., the mass-load and the mass of the girder, was also studied by other investigators among which one should mention the works by Steuding [7], Schallemcamp [8] and Bolotin [9]. A very thorough treatise on the dynamic response of several types of railway bridges crossed by steam locomotives was presented by Inglis [10], where harmonic analysis has been employed. Interesting analyses on similar bridge problems were also presented by Hilleborg [11] using Fourier's analysis and by Biggs et al. [12] using the Inglis technique. The problem of the dynamic response of bridges under moving loads was reviewed in detail by Timoshenko [13], and later on by Kolousek [14]. The extended review reported by Fryba [15] in his excellent monograph on this subject should also be mentioned. These analyses have been extended to simple frames subjected to moving loads by Karaolides and Kounadis [16] and thereafter, to a two-bar frame under a moving load in which the effect of axial motion has been taken into account [17]. Some partial results regarding the effects of the mass of a moving load on the dynamic response of a simply supported beam were presented [18].

In all the above studies, the response of vibrating beams and frames due to moving loads was established on the basis of standard dynamic analysis where the effects of centripetal force, Coriolis force and rotatory inertia, associated with the mass of the moving vehicle which follows the motion of the flexural vibrating beam has been neglected. A first study of the effect of centripetal and Coriolis forces of a single moving mass on the dynamic response of a light weighted bridge was presented through a pure analytical way in [19] by Michaltsos and Kounadis, while the effect of the above forces due to the motion of a vehicle was studied in [20]. The relatively previous existing studies used FEM and these forces are naturaly included without specific distinction, as foe example in [21].

A lot of studies followed, where the problem was analyzed through pure theoretical investigation, using finite element methods or experimental ones. Among them, one must mention the work of Majumder and Manohar [22] where the problem was studied via the finite element method to determine a possible loss of stiffness in beams caused by a moving dynamic load, the works of Dehestani et al. [23], Stancioiu et al. [24], Gonzalez et al. [25], and Nguyen and Tran [26] where several cases of beams subjected to various moving mass-loads were studied. On the other hand, Billelo et al. [27], [28] studied the problem of a bridge under a moving mass through experimental investigation. The influence of a concentrated mass-load and a vehicle moving on a bridge while bouncing on an irregularity was studied by Michaltsos [29], where the critical speeds for the vehicle's normal or not passage were also determined. Lu et al. [30] studied the frequency characteristics of a railway bridge subjected to moving trains, considering the trains' mass. Karimi and Ziaei-Rad [31] studied the problem of a beam with moving support, that is subjected to a moving mass. Dimitrovova [32] studied the influence of centripetal and Coriolis forces on a beam on elastic foundation.

The present work deals with the linear dynamic response of a simply supported light-weighted (steel) bridge with negative sag under moving concentrated or distributed mass-loads with constant magnitude and velocity. This analysis focuses on the determination of the influence limits of both the concentrated and the distributed mass-loads on the bridges' dynamic response with respect to their velocity and to the sag of the bridge. The individual and coupling effect of these parameters on the dynamic response of the bridge are thoroughly discussed herein A 2D model is considered for the solution of the bridge, while the theoretical formulation is based on a continuum approach, which has been used in the literature to analyze such bridges. A variety of numerical examples allow one to draw important conclusions for structural design purposes.

2 MATHEMATICAL FORMULATION

2.1 Basic definitions and assumptions

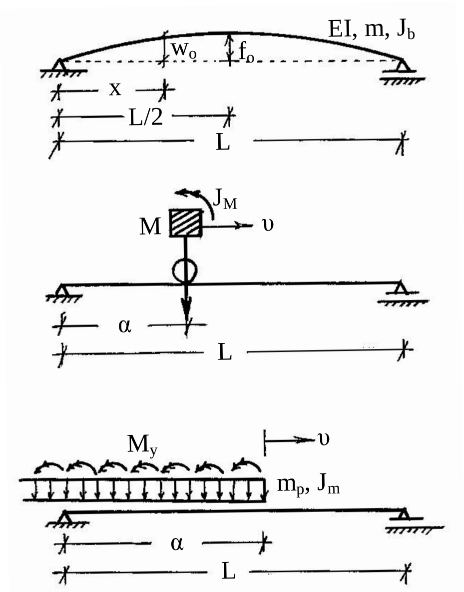


Figure 1. Geometry and sign convention of a simply supported beam under moving mass-loadings

Let us consider the simply supported beam shown in Fig. 1, having a prismatic cross-section with constant mass per unit length m, flexural rigidity EI, and corresponding rotatory inertia J_M, made from linear, homogeneous and isotropic material and having an initial sag w_o, which usually can be expressed by the parabola:

$$w_o(x) = -\frac{4f_o}{L^2} \cdot x^2 + \frac{4f_o}{L} \cdot x \qquad (1)$$

where f_o is the sag magnitude at $x = L/2$, while it is entered in equation (1) with its sign.

We note here that $w_o(x)$ is smaller than L/200 and therefore it is not necessary to follow the theory of curved beams.

At $\mathbf{t = 0}$ either a concentrated or a distributed mass-load enters the beam, moving with constant speed υ and having mass M and rotatory inertia J_M (concentrated mass-load) or mass m and J_m (distributed mass-load).

2.2 The concentrated mass-load case

These forces acting at the point x=a of the beam are the following [33] comprehensively depicted in figure 2:

a. The external gravitational load $F = M \cdot g$ (2a)

b. The external motive force $P(t)$ (2b)

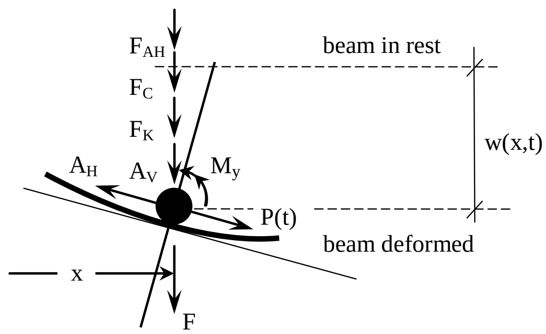


Figure 2. Analysis of the forces developing due to the traveling mass-load F=mg

c. The vertical inertia force, being equal to: $A_V = -M \cdot [\ddot{w}(\alpha, t) + \ddot{w}_o(\alpha, t)]$ (2c)

d. The centripetal force which is given by:

$$F_K = -M \cdot \frac{\upsilon^2}{R} = -M \cdot \dot{s}^2 \cdot \frac{\partial^2 (w + w_o)}{\partial x^2} = -M \cdot \dot{s}^2 \cdot [w''(\alpha, t) + w_o''(\alpha)] \qquad (2d)$$

e. The Coriolis force which can be written as follows:

$$F_C = -2 \cdot M \cdot \dot{s} \cdot \frac{\partial^2 (w + w_o)}{\partial x \, \partial t} = -M \cdot \dot{s} \cdot [\dot{w}'(\alpha, t) + \dot{w}_o'(\alpha)] \qquad (2e)$$

f. The projection of the tangential inertia force A_H on the vertical axis:

$$F_{AH} = -M \cdot \ddot{s} \cdot (w + w_o)' = -M \cdot \ddot{s} \cdot [w'(\alpha, t) + w_o'(\alpha)] \qquad (2f)$$

g. The moment M_y due to the rotator inertia of mass-load, given by the relation:

$$M_y = J_M[\ddot{w}'(\alpha, t) + \ddot{w}'_o(\alpha)] \tag{2g}$$

Neglecting the longitudinal deformations, the differential equations of the forced flexural vibrations of the bridge are given by the following equations:

$$\begin{aligned} EIw'''' + c\dot{w} + m\ddot{w} - J_b\ddot{w}'' &= Mg\,\delta(x-\alpha) - \\ &- M[\ddot{w} + \dot{s}^2(w'' + w''_o) + 2\dot{s}\,\dot{w}' + \ddot{s}(w' + w'_o)]\delta(x-\alpha) + \\ &+ J_M\ddot{w}'\delta'(x-\alpha) \end{aligned} \tag{3a}$$

$$M[\ddot{s} + \ddot{w}\,w' + \dot{w}\,\dot{w}']\delta(x-\alpha) = P(t) \tag{3b}$$

Taking into account that because of eq (1) it is: $w''''_o = \dot{w}_o = \ddot{w}_o = 0$, and that for constant velocities it is $\dot{\mathbf{s}} = \upsilon$, $\ddot{\mathbf{s}} = \mathbf{0}$ the first of equations (3) can be rewritten as follows:

$$\begin{aligned} EIw'''' + c\dot{w} + m\ddot{w} - J_b\ddot{w}'' = Mg\,\delta(x-\alpha) - M[\ddot{w} + \upsilon^2(w'' + w''_o) \\ + 2\upsilon\dot{w}']\delta(x-\alpha) + J_M\ddot{w}'\delta'(x-\alpha) \end{aligned} \tag{4}$$

A series solution of eq(4) in terms of normal modes can be sought in the form:

$$w(x,t) = \sum_n X_n(x)\cdot T_n(t) \tag{5}$$

where $X_n(x)$ are the shape functions of the freely vibrating bridge, while $T_n(t)$ are the corresponding time functions which have to be determined.
Introducing the last expression of w(x,t) into eq(4), one obtains:

$$\left.\begin{aligned} &EI\sum_n X''''_n T_n + c\sum_n X_n\dot{T}_n + m\sum_n X_n\ddot{T}_n - J_b\sum_n X''_n\ddot{T}_n = \\ &= Mg\,\delta(x-\alpha) - M\left[\sum_n X_n\ddot{T}_n + \upsilon^2\sum_n X''_n T_n - \upsilon^2\cdot\frac{8f_o}{L^2} + 2\upsilon\sum_n X'_n\dot{T}_n\right]\delta(x-\alpha) + \\ &\qquad + J_M\sum_n X'_n\ddot{T}_n\cdot\delta'(x-\alpha) \end{aligned}\right\} \tag{6a}$$

The freely vibrating bridge is governed by the following equation:

$$\left.\begin{aligned} &EI\sum_n X''''T_n - m\sum_n \omega_n^2 X_n T_n = 0, \\ &\text{where}\quad X_n = sin\frac{n\pi x}{L},\quad \omega_n^2 = \frac{n^4\pi^4 EI}{mL^4}, n = 1, 2, \ldots. \end{aligned}\right\} \tag{6b}$$

Because of eq(7b), eq (7a) becomes:

$$\left.\begin{aligned}&m\sum_n X_n\ddot{T}_n + c\sum_n X_n\dot{T}_n + m\sum_n \omega_n^2 X_n = \\ &= Mg\,\delta(x-\alpha) - M\left[\sum_n X_n\ddot{T}_n + \upsilon^2\sum_n X_n''T_n - \upsilon^2\cdot\frac{8f_o}{L^2} + 2\upsilon\sum_n X_n'\dot{T}_n\right]\delta(x-\alpha) + \\ &\qquad + J_M\sum_n X_n'\ddot{T}_n\cdot\delta'(x-\alpha) + J_b\sum_n X_n''\ddot{T}_n\end{aligned}\right\}\quad (6c)$$

Multiplying eq (6c) by X_k, $(k \neq n)$ and integrating from 0 to L one obtains:

$$\left.\begin{aligned}&\ddot{T}_k + \frac{c}{m}\dot{T}_k + \omega_k^2 T_k = \\ &= \frac{2Mg}{mL}\cdot X_k(\alpha) - \frac{2M}{mL}X_k(\alpha)\left[\sum_n X_n(\alpha)\ddot{T}_n + \upsilon^2\sum_n X_n''(\alpha)T_n - \frac{8f_o\upsilon^2}{L^2} + 2\upsilon\sum_n X_n'(\alpha)\dot{T}_n\right] - \\ &\quad - \frac{J_M}{mL}\left[X_k(\alpha)\sum_n X_n''(\alpha)\ddot{T}_n + X_k'(\alpha)\sum_n X_n'(\alpha)\ddot{T}_n\right] + \frac{2J_b}{mL}\int_0^L\left(X_k(x)\sum_n X_n''(x)\ddot{T}_n\right)dx\end{aligned}\right\}\quad (6d)$$

Clearly, a closed form solution of equation (6d) is not possible. However, one can seek approximate solutions, based on previous work [20], [34].

A first approximate solution of eq(6d) is obtained by considering as loading the force Mg and by ignoring the damping term. This leads to:

$$\left.\begin{aligned}&\overline{T}_n(t) = \frac{2Mg}{mL(\omega_n^2 - \Omega_n^2)}\left(\sin\Omega_n t - \frac{\Omega_n}{\omega_n}\sin\omega_n t\right)\\ &\text{with: } \quad \Omega_n = \frac{n\pi\upsilon}{L}\end{aligned}\right\}\quad (6e)$$

Introducing the last expression into the right side of equation (6d), we obtain:

$$\left.\begin{aligned}&\ddot{T}_k + \frac{c}{m}\dot{T}_k + \omega_k^2 T_k = F_k(t), \quad \text{where}\\ &F_k(t) = \frac{2Mg}{mL}\cdot X_k(\upsilon t)\\ &\quad - \frac{2M}{mL}X_k(\upsilon t)\left[\sum_n X_n(\upsilon t)\ddot{T}_n + \upsilon^2\sum_n X_n''(\upsilon t)T_n - \frac{8f_o\upsilon^2}{L^2} + 2\upsilon\sum_n X_n'(\upsilon t)\dot{T}_n\right] - \\ &\quad - \frac{J_M}{mL}\left[X_k(\upsilon t)\sum_n X_n''(\upsilon t)\ddot{T}_n + X_k'(\upsilon t)\sum_n X_n'(\upsilon t)\ddot{T}_n\right] + \frac{2J_b}{mL}\int_0^L\left(X_k(x)\sum_n X_n''(x)\ddot{T}_n\right)dx\end{aligned}\right\}\quad (7)$$

The solution of eq(7), with initial conditions $w(x,0) = \dot{w}(x,0) = 0$ is given by the Duhamel's integral:

$$\left.\begin{aligned} T_k(t) &= \frac{1}{\overline{\omega}_k} \cdot \int_0^t F_k(\tau) \cdot e^{-\beta(t-\tau)} \sin \overline{\omega}_k (t-\tau)\, d\tau \\ \text{where:} \quad \beta &= \frac{c}{2m}, \quad \overline{\omega}_k^2 = \omega_k^2 - \beta^2 \end{aligned}\right\} \tag{8}$$

2.3 The distributed mass-load case

For the loading situation shown in Fig. 1c, one may find that:

$$\sum_{i=1}^{\lambda} P_i \delta(x-\alpha_i) = \underbrace{\lim}_{\Delta x \to 0} \left(\frac{P_i}{\Delta x} \delta(x-\alpha_i)\, \Delta x \right) = \int_0^{\alpha} p(x)\, \delta(x-\alpha_i)\, dx = p(\alpha_i) = m \cdot g \tag{9a}$$

as well as:

$$\sum_{i=1}^{\lambda} M_y \delta'(x-\alpha_i) = \underbrace{\lim}_{\Delta x \to 0} \left(\frac{M_y}{\Delta x} \delta'(x-\alpha_i)\, \Delta x \right) = \int_0^{\alpha} \frac{J_M \ddot{w}'(x)}{\Delta x} \delta'(x-\alpha_i)\, dx = -J_M \ddot{w}''(\alpha_i) \tag{9b}$$

Substituting expressions (10a) and (10b) into eq (5) one gets:

$$EIw'''' + c\dot{w} + m\ddot{w} - J_b \ddot{w}'' = m_p g - m_p [\ddot{w} + \upsilon^2 (w'' + w_o'') + 2\upsilon \dot{w}'] - J_M \ddot{w}'' \tag{9c}$$

A series solution of eq(9c) in terms of normal modes can be sought in the form:

$$w(x,t) = \sum_n X_n(x) \cdot P_n(t) \tag{10a}$$

where $X_n(x)$ are the shape functions of the freely vibrating bridge, while $P_n(t)$ are the corresponding time functions to be determined.

Introducing eq(10a) into eq(9c) and because of eq(1) one gets:

$$\left.\begin{aligned} &EI\sum_n X_n'''' P_n + c\sum_n X_n \dot{P}_n + m\sum_n X_n \ddot{P}_n - J_b \sum_n X_n'' \ddot{P}_n = \\ &= m_p g - m_p \sum_n X_n \ddot{P}_n - m_p \upsilon^2 \sum_n X_n'' P_n + m_p \upsilon^2 \frac{8 f_o}{L^2} - 2 m_p \upsilon \sum_n X_n' \dot{P}_n - J_M \sum_n X_n'' \ddot{P}_n \end{aligned}\right\} \tag{10b}$$

Because of eq(6a), the above equation becomes:

$$\left.\begin{aligned} &m\sum_n X' \ddot{P}_n + c\sum_n X_n \dot{P}_n + m\sum_n \omega_n^2 X_n P_n - J_b \sum_n X_n'' \ddot{P}_n = \\ &= m_p g - m_p \sum_n X_n \ddot{P}_n - m_p \upsilon^2 \sum_n X_n'' P_n + m_p \upsilon^2 \frac{8 f_o}{L^2} - 2 m_p \upsilon \sum_n X_n' \dot{P}_n - J_M \sum_n X_n'' \ddot{P}_n \end{aligned}\right\} \tag{10c}$$

Multiplying the above by X_k and integrating from 0 to L one obtains:

$$\left.\begin{aligned}
&\ddot{P}_k + \frac{c}{m}\dot{P}_k + \omega_k^2 P_k = \frac{2m_p g}{mL}\cdot\int_0^{\alpha} X_k dx - \\
&-\frac{2m_p}{mL}\left\{\int_0^{\alpha}\sum_n X_n X_k \ddot{P}_n dx + \upsilon^2\int_0^{\alpha}\sum_n X_n'' X_k P_n dx - \frac{8f_o\upsilon^2}{L^2}\int_0^{\alpha} X_k dx + 2\upsilon\int_0^{\alpha}\sum_n X_n' X_k \dot{P}_n dx\right\} \\
&-\frac{2J_M}{mL}\int_0^{\alpha}\sum_n X_n'' X_k \ddot{P}_n dx + \frac{2J_b}{mL}\int_0^{L}\sum_n X_n'' X_k \ddot{P}_n dx
\end{aligned}\right\} \quad (10d)$$

Following the procedure of §2, we consider as a first approximate function of $P_n(t)$ the one corresponding to the dynamic response of the beam under a moving distributed load p(x) with constant speed υ and without mass, as follows:

$$\left.\begin{aligned}
&\overline{P}_n(t) = \frac{2m_p g}{n m \pi \omega_n}\cdot\left\{\frac{1-\cos\omega_n t}{\omega_n} + \frac{\omega_n}{\Omega_n^2-\omega_n^2}(\cos\Omega_n t - \cos\omega_n t)\right\} \\
&\text{with:}\quad \Omega_n = \frac{n\pi\upsilon}{L}
\end{aligned}\right\} \quad (11)$$

Introducing the last expression into the right side of eq (10d) and taking into account that $\alpha = \upsilon t$ one obtains:

$$\left.\begin{aligned}
&\ddot{P}_k + \frac{c}{m}\dot{P}_k + \omega_k^2 P_k = G_k(t)\,, \quad \text{where:} \\
&G_k(t) = \frac{2m_p g}{mL}\cdot\int_0^{\upsilon t} X_k dx - \\
&\quad -\frac{2m_p}{mL}\left\{\int_0^{\upsilon t}\sum_n X_n X_k \ddot{\overline{P}}_n dx + \upsilon^2\int_0^{\upsilon t}\sum_n X_n'' X_k \overline{P}_n dx - \frac{g f_o\upsilon^2}{L^2}\int_0^{\upsilon t} X_k dx + 2\upsilon\int_0^{\upsilon t}\sum_n X_n' X_k \dot{\overline{P}}_n dx\right\} \\
&\quad -\frac{2J_M}{mL}\int_0^{\upsilon t}\sum_n X_n'' X_k \ddot{\overline{P}}_n dx + \frac{2J_b}{mL}\int_0^{L}\sum_n X_n'' X_k \ddot{\overline{P}}_n dx
\end{aligned}\right\} \quad (12)$$

The solution of the first of eqs (12) with initial conditions $w(x,0) = \dot{w}(x,0) = 0$, is given by the Duhamel's integral:

$$\left.\begin{aligned}
&P_k(t) = \frac{1}{\overline{\omega}_k}\cdot\int_0^{t} G_k(\tau)\cdot e^{-\beta(t-\tau)}\sin\overline{\omega}_k(t-\tau)\,d\tau \\
&\text{where:}\quad \beta = \frac{c}{2m}\,, \quad \overline{\omega}_k^2 = \omega_k^2 - \beta^2
\end{aligned}\right\} \quad (13)$$

3 NUMERICAL RESULTS AND DISCUSSION

Given that the influence of the Centripetal and Coriolis forces caused by a moving concentrated mass-load or a biaxial vehicle have been studied in detail through an analytical way firstly in [19], [20] and then by other investigators, the purpose of this paper is: a) to study the effect of the sag, the centripetal and the Coriolis forces on the dynamic behavior of the bridge due to the action of a concentrated mass-load moving with constant velocity, b) to study the same effects for a distributed mass-load moving with constant velocity, and c) to present a comparison of the above effects and determine the range for each case of loading where each effect should be taken into account in relation to the sag of bridge.

More specifically, a simply supported bridge of length equal to $L = 100\mathrm{m}$ is chosen with mass per unit length $m = 400\,\mathrm{kg/m}$, moment of inertia $I_y = 1.0\,\mathrm{m}^4$ and rotatory inertia per unit length $J_b = 4\,\mathrm{kN\,m}\,sec^2/\mathrm{m}$, representing a light-weighted bridge. The bridge is made from structural steel (isotropic and homogeneous material) with modulus of elasticity $E = 2.1\cdot 10^8\,\mathrm{kN/m^2}$ having a slight curvature with sag at its middle equal to $f_o = -L/500$ or $f_o = -L/200$.

3.1 The concentrated mass-load

Let us consider now a concentrated mass-load, having a ratio M/mL as it is shown in Table 1 and a fixed rotatory inertia $J_M = 10\,\mathrm{kN\,m\,sec}^2$.

Table 1. Dynamic deformations due to a concentrated mass-load

M/mL		0.2		0.4		0.6		0.8	
v (m/s) km/h	Loading	f_o	w	f_o	w	f_o	w	f_o	w
(20) 72	OW	0	0.0885	0	0.1962	0	0.3235	0	0.4703
	OWC	0	0.0886	0	0.1971	0	0.3259	0	0.4748
	OWC1	-L/500	0.0881	-L/500	0.1961	-L/500	0.3243	-L/500	0.4728
	OWC2	-L/200	0.0873	-L/200	0.1946	-L/200	0.3221	-L/200	0.4697
	D1%		0.132		0.461		0.734		0.961
	D2%		-0.569		-0.510		-0.462		-0.422
	D3%		-1.423		-1.275		-1.156		-1.057
(40) 144	OW	0	0.0888	0	0.2064	0	0.3572	0	0.5416
	OWC	0	0.0884	0	0.2060	0	0.3573	0	0.5424
	OWC1	-L/500	0.0864	-L/500	0.2023	-L/500	0.3519	-L/500	0.5353
	OWC2	-L/200	0.0836	-L/200	0.1969	-L/200	0.3438	-L/200	0.5249
	D1%		-0.465		-0.185		0.020		0.165
	D2%		-2.193		-1.782		-1.507		-1.308
	D3%		-5.470		-4.450		-3.764		-3.267
(60) 216	OW	0	0.1058	0	0.2389	0	0.4032	0	0.5995
	OWC	0	0.1078	0	0.2551	0	0.4468	0	0.6833
	OWC1	-L/500	0.1027	-L/500	0.2452	-L/500	0.4323	-L/500	0.6642
	OWC2	-L/200	0.0949	-L/200	0.2305	-L/200	0.4107	-L/200	0.6357
	D1%		1.900		6.764		10.80		13.98
	D2%		-4.812		-3.867		-3.239		-2.791
	D3%		-11.98		-9.641		-8.082		-6.968

At $t = 0$, where the bridge is at rest, the concentrated mass-load enters the bridge with constant velocity υ. The studied velocities of the moving load vary from $20\,m/sec$ $(72\,km/h)$ to $60\,m/sec$ $(216\,km/h)$, which corresponds from a slow to a super fast train.

Three cases of loading have been studied. The first one is denoted by OW, where is taken into account only the influence of mass M of the load. The second case is denoted by OWC, where the influence of the mass of the load and the Centripetal and Coriolis forces are taken into account. The third and fourth load cases denoted by OWC1 and OWC2 are similar to the OWC case where initial sag at mid-length $f_o = -L/500$ and $f_o = -L/200$ are also included in the analysis.

In addition, in Table 1 the following percentages are shown:

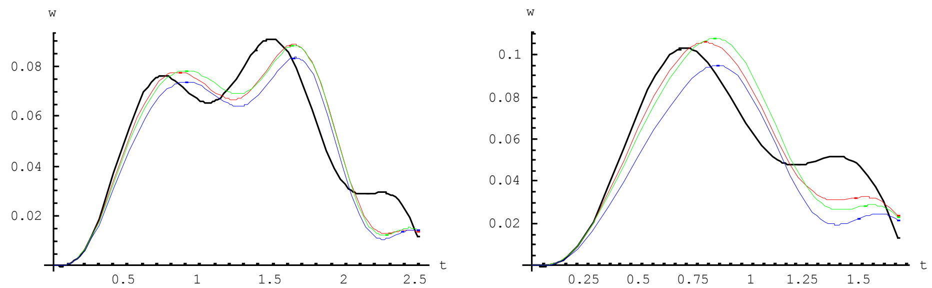
$$D1 = \frac{OWC - OW}{OW}\%, \quad D2 = \frac{OWC1 - OWC}{OWC}\%, \quad D2 = \frac{OWC2 - OWC}{OWC}\%.$$

The results of the above study are tabulated in the following Table 1.

From the above table the notable influence of Centripetal and Coriolis forces is verified once again, which increases the dynamic deformations of the bridge from ~0.13% for low speeds and values of M/mL up to ~14% for high speeds and high values of M/mL.

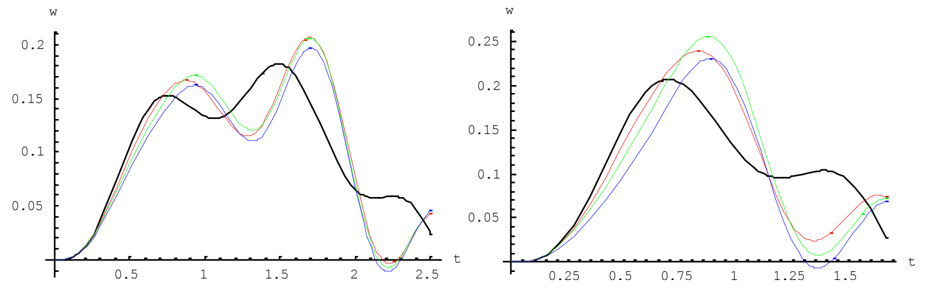
On the other hand, one can see that a negative curvature decreases the deformations from ~0.60% to ~12% for low speeds and values of M/mL, while this decrease is less than the previous case, ranging from 0.40% to 7% for high speeds and values of M/mL.

Characteristic plots of the dynamic response of the bridge are shown in Figs 3 to 6.



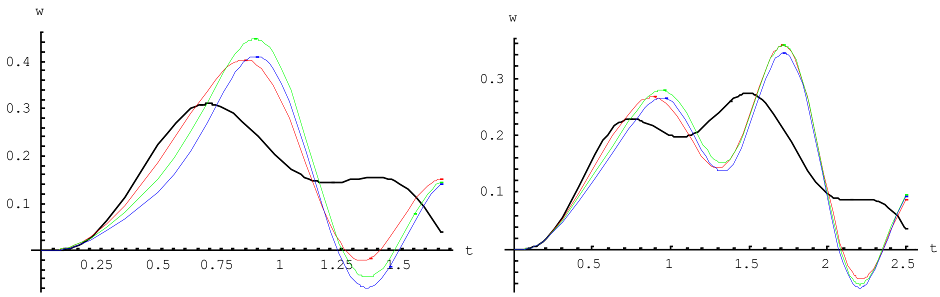
a. υ=144 km/h b. υ=216 km/h

Figure 3. Dynamic deformations of the middle of the bridge for M/mL=0.2, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue



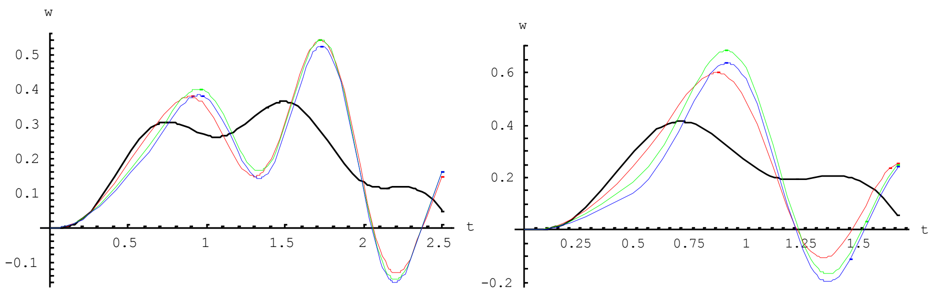
a. υ=144 km/h b. υ=216 km/h

Figure 4. Dynamic deformations of the middle of the bridge for M/mL=0.4, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue



a. υ=144 km/h b. υ=216 km/h

Figure 5. Dynamic deformations of the middle of the bridge for M/mL=0.6, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue



a υ=144 km/h b. υ=216 km/h

Figure 6. Dynamic deformations of the middle of the bridge for M/mL=0.8, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue

3.2 The distributed mass-load

Let us consider now a distributed mass-load, having a ratio $m_p / m \cdot L$ as it is shown in Table 2 and fixed rotatory inertia $J_m = 1 \mathrm{kN\,m\,sec}^2$.

At $t = 0$, where the bridge is at rest, the distributed mass-load enters the bridge with constant velocity υ.

The studied velocities of the moving load, the cases of loading and percentages are the same as in §3.1 and the obtained results of this study are shown in Table 2. The obtained results from the above study, shown in Table 2, concerning the passage of a distributed mass-load, are quite different from those for the passage of a concentrated mass- load. The Centripetal and Coriolis forces have negligible influence on the dynamic response of a steel bridge regardless of the ratio m_p/m and the velocity of the moving load.

Table 2. Dynamic deformations due to distributed mass-load

m_p/m		0.1		0.2		0.3		0.4	
v (m/s) km/h	Loading	f_o	w	f_o	w	f_o	w	f_o	w
(20) 72	OW	0	0.0310	0	0.0744	0	0.1300	0	0.1981
	OWC	0	0.0295	0	0.0714	0	0.1257	0	0.1923
	OWC1	-L/500	0.0294	-L/500	0.0711	-L/500	0.1252	-L/500	0.1917
	OWC2	-L/200	0.0291	-L/200	0.0706	-L/200	0.1245	-L/200	0.1907
	D1%		-4.837		-3.984		-3.373		-2.914
	D2%		-0.538		-0.445		-0.379		-0.331
	D3%		-1.347		-1.114		-0.949		-0.828
(40) 144	OW	0	0.0289	0	0.0656	0	0.1101	0	0.1622
	OWC	0	0.0281	0	0.0643	0	0.1084	0	0.1606
	OWC1	-L/500	0.0275	-L/500	0.0630	-L/500	0.1065	-L/500	0.1581
	OWC2	-L/200	0.0265	-L/200	0.0611	-L/200	0.1036	-L/200	0.1542
	D1%		-2.678		-1.977		-1,409		-0.938
	D2%		-2.283		-1.998		-1.776		-1.599
	D3%		-5.708		-4.995		-4.440		-3.997
(60) 216	OW	0	0.0282	0	0.0616	0	0.1007	0	0.1453
	OWC	0	0.0281	0	0.0622	0	0.1026	0	0.1493
	OWC1	-L/500	0.0266	-L/500	0.0592	-L/500	0.0981	-L/500	0.1434
	OWC2	-L/200	0.0244	-L/200	0.0548	-L/200	0.0916	-L/200	0.1346
	D1%		-0.401		0.803		1.855		2.751
	D2%		-5.280		-4.734		-4.292		-3.926
	D3%		-13.19		-11.83		-10.73		-9.814

In some cases, these mass-forces are mitigating to the dynamic response of the bridge. In any case their influence is very little ranging from -2% to 1%.

Only for very heavy trains crossing the bridge with high speeds (>200 km/h), one can observe an enhanced influence of the dynamic response of the bridge, which in any case varies from about 0.8% to about 4%.

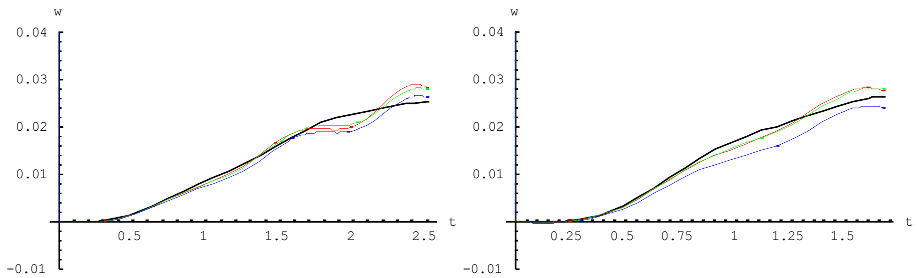
The negative curvature that is applied for this loading case, decreases the deformations:

from 1.3% (low speeds) to 14% (high speeds) for the ratio m_p/m=0.1,

from 1.1% (low speeds) to 12% (high speeds) for the ratio m_p/m=0.2,

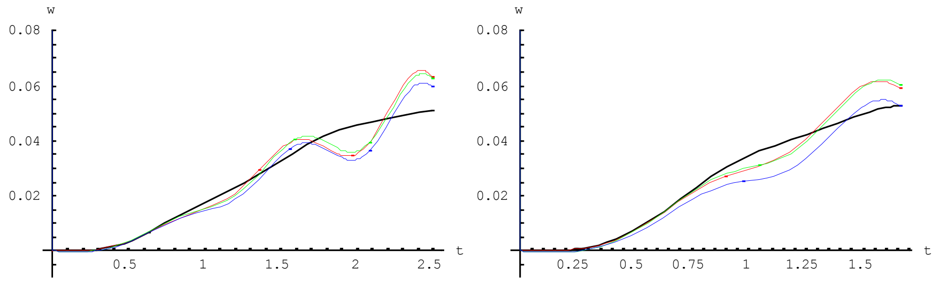
from 0.9% (low speeds) to 11% (high speeds) for the ratio $m_p/m=0.3$,
from 0.8% (low speeds) to 9.5% (high speeds) for the ratio $m_p/m=0.4$.

Furthermore, it is observed that the small curvatures –L/500 produce also notable decrease of the deformation.
Characteristic plots are shown in Figures 7 to 10.



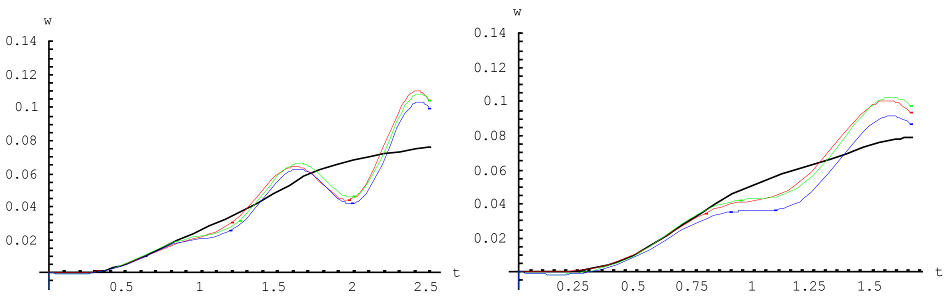
a. υ=144 km/h b. υ=216 km/h

Figure 7. Dynamic deformations of the middle of the bridge for $m_p/m=0.1$, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue



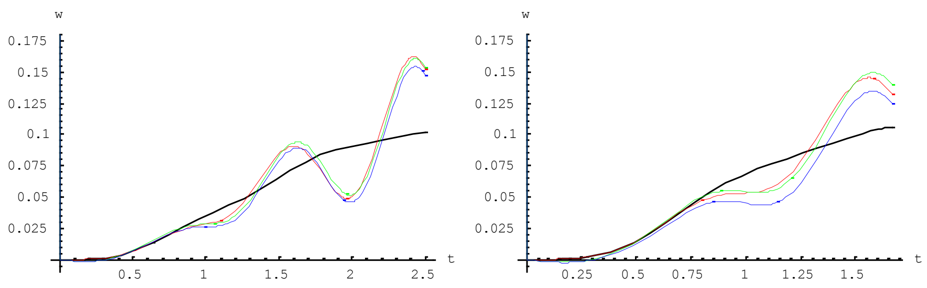
a. υ=144 km/h b. υ=216 km/h

Figure 8. Dynamic deformations of the middle of the bridge for $m_p/m=0.2$, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue



a. υ=144 km/h b. υ=216 km/h

Figure 9. Dynamic deformations of the middle of the bridge for $m_p/m=0.3$, f_o=--L/200 without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue



a. υ=144 km/h b. υ=216 km/h

Figure 10. Dynamic deformations of the middle of the bridge for $m_p/m=0.4$, $f_o=-L/200$ without mass forces: black, OW: red, OW+OWC: green, OW+OWC+OWC2: blue

4 CONCLUSIONS

From the results of the bridge model considered herein, one can draw the following conclusions:

- For the case of a concentrated mass-load, it is verified that the influence of the mass-forces is significant and that one has to take them into account. Besides the influence of the inertia force of the moving mass, one must add the Centripetal and Coriolis forces as well as the rotator inertia of the mass-load that have an additional influence ranging from 0.4% to 15%. A small negative sag $f_o = -L/500$ decreases the dynamic deformations from 0.5% to 4%, while a bigger sag $f_o = -L/200$ decreases the dynamic deformations from 1.5% to 12%. The value of the last sag for the studied bridge of length $L = 100\,m$ is $f_o = 0.5\,m$. It is clear that one can build bridges with bigger curvatures, taking always into account the necessary convenience of the passing vehicles.
- For the case of a distributed mass-load, it is observed that the most significant term, which has the biggest influence on the dynamic behavior of the bridge, is the one expressing the inertia forces of the distributed mass-load, while the terms expressing the Centripetal and Coriolis forces do not cause significant additional dynamic deformations. Indeed, for a wide range of moving masses and velocities their presence is rather relieving. Only for extreme values ($m_p / m > 0.4$ and $\upsilon > 200\,km/h$) the loading with distributed mass-loads shows an increase in dynamic deformations in the order of about 4%. The curvatures, for this kind of loads, significantly decrease the dynamic deformations of the bridge. This reduction amounts from to ~0.3% to ~4% for $f_o = -L/500$ and from to ~1.0% to ~14% for $f_o = -L/200$.
- It is interesting to explain why the bridge presents such a response under the action of concentrated mass-load which is quite different than the response

under the action of distributed mass-load. Indeed, one can observe that in the case of a distributed moving mass-load, this type of loading does not cause an increase of the dynamic deformations of the bridge (such as the concentrated mass-load), but on the contrary it causes as slight decrease. In order to investigate this phenomenon one can observe the plots of figures 3 to 6, where it is shown that when the concentrated mass-load moves on the first half of the bridge the Centripetal and Coriolis forces cause a decrease of the bridge's deformations regardless of mass magnitude and velocity, and only when the load enters the second half the above forces start causing an increase of the bridge's deformations. Probably, this is due to the fact that when the distributed moving mass-load enters the second half span, the first half is still loaded and participating in the movement of the bridge.

REFERENCES

1. Willis R. Preliminary essay to the appendix B: experiments for determining the effects produced by causing weights to travel over bars with different velocities. In: Report of the Commissioners Appointed to Inquire into the Application of Iron to Railway Structures (G. Crey et al. editors), W. Clowes and Sons, London, 1849.
2. Stokes G. Discussion of a differential equation relating to the breaking of railway bridges. Mathematical and Physical Papers 1883;178-220.
3. Zimmermann H. Die Schwingungen eines Tragers mit bewegter Lasts. Centalblatt der Bauverwaltung 1896;16:249-251,257-260,264-266,288.
4. Krylov AN. Uber die erzwungenen Schwingungen von gleichformigen elastischen Staben. Mathematical Collection of Papers of the Academy of Sciences, Petersburg 1905;61.
5. Timoshenko SP. Erzwungene Schwingungen prismatischer Stabe. Zeitschrift. Fuer. Mathematik und Physik 1911;59:163-203.
6. Lowan AN. On transverse oscillations of beams under the action of moving variable loads. Philisophical Magazine 1935;7-19:708-715.
7. Steuding H. Die Schingungen von tragern bei bewegten lasten. I, II. Ingenieur-Archiv 1934;275-305,265-270.
8. Schallenkamp A. Schwingungen von tragern bei bewegten lasten. Ingenieur-Archiv 1937;8:182-198.
9. Bolotin VV. Problem of bridge vibration under the action of a moving load (in Russian). Izvestiya AN SSSR, Mekhanika I Mashinostroenie 1961:**4:**109-115.
10. Inglis CE. A Mathematical Treatise on Vibration in Railway Bridges. The University Press, Cambridge, 1934.
11. Hillerborg A. Dynamic influences of smoothly running loads on simply supported girders. Kungl. Tekn. Hogskolan, Stockholm, 1951.
12. Biggs JM, Suer HS, Louw JM. Vibration on simple-span highway bridges. Transactions of the American Society of Civil Engineers 1959;124:291-318.
13. Timoshenko SP. History of the Strength of Materials. D. van Nostrand Co, New York, 1953.
14. Kolousek V. Dynamics of civil engineering structures. Part I-General problems, second edition; Part II-Continuous beams and frame systems, second edition; Part III-Selected topics (in Czech). SNTL Prague 1956, 1961, 1967.
15. Fryba L, Vibrations of Solids and Structures under Moving Loads. Research Institute of Transport. Prague 1972.
16. Karaolides C, Kounadis AN. Forced motion of a simple frame subjected to a moving force. Journal of Sound and Vibration 1983;88:37-45.

17. Sophianopoulos DS, Kounadis AN. The axial motion effect on the dynamic response of a laterally vibrating frame subject to a moving load. Acta Mechanica 1989;79:277-294.
18. Michaltsos GT, Sophianopoulos DS, Kounadis AN. The effect of a moving mass and other parameters on the dynamic response of a simply supported beam. Journal of Sound and Vibration 1996;191:357-362.
19. Michaltsos GT, Kounadis AN. The effects of centripetal and Coriolis forces on the dynamic response of light bridges under moving loads. Journal of Vibration and Control 2001;**7:**315-326.
20. Michaltsos GT. The influence of centripetal and Coriolis forces on the dynamic response of light bridges under moving vehicles. Journal of Sound and Vibration 2001;247(2):261-277.
21. Olsson M., Finite element, modal co-ordinate analysis of structures subjected to moving loads. J. of Sound and Vibration 1985, 99(1):1-12
22. Majumder L, Manohar CS. Nonlinear reduced models for beam damage detection using data on moving oscillator-beam interactions. Computers and Structures 2004;82:301-314.
23. Dehestani M, Mofit M, Vafai A. Investigation of critical influential speed for moving mass problems on beams. Applied Mathematical Modelling, 2009;33:3885-3895.
24. Stancioiu D, James S, Quyang H, Mottershed JE. Journal of Physics: Conference Series, 2009;181.
25. Gonzalez A, O'Brien EJ, Mc Getric P. Detection of bridge dynamics parameters using instrumental vehicle. In: Proc. of 5th World Conference on Structural Control and Monitoring, Tokyo 2010.
26. Nguyen KV, Tran HT. Multi-cracks detection of a beam like structure based on the on-vehicle vibration signal and wavelet analysis. Journal of Sound and Vibration 2010;329:4455-4465.
27. Bilello C, Bergman LA, K$_{UCHMA}$ D. Experimental investigation of a small-scale bridge model under a moving mass. ASCE Journal of Structural Engineering 2004;130:799-804.
28. Bilello C, Bergman LA. Vibration of damaged beams under a moving mass: Theory and experimental validation. Journal of Sound and Vibration 2004;274:567-582.
29. Michaltsos GT. Bouncing of a vehicle on an irregularity – A mathematical model. Journal of Vibration and Control 2010;16(2):181-206.
30. Lu Y, Lei M, Woodward P. Frequency characteristics of railway bridge response to moving trains with consideration of trains mass. Engineering Structures 2012; 42:9-22.
31. Karimi AH, Ziaei-Rad S. Vibration analysis of a beam with moving support subjected to a moving mass travelling with constant and variable speed. Communications in Nonlinear Science and Numerical Simulation 2015;29:372-390.
32. Dimitrovova Z., New semi-analytical solution for a uniformly moving mass on a beam on a two- parameter visco-elastic foundation. Int J of Mechanical Sciences 2017;127:142-162.
33. Michaltsos GT, Raftoyiannis IG. Bridges' Dynamics, Bentham e-books, UAE, 2011. e-ISBN: 978-1-60805-220-2
34. Kounadis AN. An efficient and simple approximate technique for solving non linear initial and boundary problems. Comp. Mechanics 1999; 9: 221-231.