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THE EXACT DYNAMIC CHARACTERISTICS AND DEFORMATION OF SINGLE SPAN BRIDGES UNDER THE ACTION OF A MOVING MASS-LOAD

Ioannis G. Raftoyiannis[1] and George T. Michaltsos[2]

[1,2] National Technical University of Athens, Dept. of Civil Engineering, Greece
e-mail: rafto@central.ntua.gr, michalts@central.ntua.gr

ABSTRACT: This paper deals with the influence of a moving mass-load on the eigenfrequencies of a single span bridge as well as its dynamic deformations taking into account the above altered eigenfrequencies that are constantly changing due to the mass-load movement. A simple but efficient 2-DOF model is considered in order to study the dynamic behavior of the bridge, while the theoretical formulation is based on the modal superposition approach and the continuum approach, which has been widely used in the bibliography to analyze such problems. The resulting non-linear differential system of equations of motion is solved with the aid of a commercial symbolic manipulator and useful results are gathered and presented in graphical and tabular form

KEYWORDS: Beam dynamics; Bridges; Moving load; Moving mass; Altered eigenfrequencies.

1 INTRODUCTION

A lot of research has been reported during the last 100 years dealing with the dynamic response of railway bridges and later of highway bridges, under the influence of moving loads. Extensive references to the literature on this subject can be found in the excellent book of Frýba [1].

Two early contributions in this area by Stokes [2] and Zimmerman [3] set the background for dealing with the problem of moving loads. Krýlov [4] presented a complete solution to the problem of the dynamic behavior of a prismatic bar under a constant magnitude load moving with constant velocity. Timoshenko [5] solved the same problem, but for a harmonic pulsating moving load. Another pioneer work on this subject was presented by Inglis [6], in which numerous parameters were taken into account. Hillerborg [7] gave an analytical solution to the previous problem by means of Fourier's method.

Despite the availability of powerful computers, most of the methods used today for analyzing bridge vibration problems are essentially based on the early techniques of Inglis or Hillerborg. Relevant publications in this area of study are the ones by Saller [8], Jeffcot [9], Steuding[10], Honda *et al* [11], Gillespi [12], Green & Cebon [13], Green *et al* [15], Zibdeh & Reckwitz [16], Lee [17], Michaltsos *et al* [18], Xu *et al* [19] , Foda & Abduljabbar [20] and Michaltsos [21, 22].

Despite of the large number of studies for over 50 years, bridges as well as other structures subjected to dynamic loads have been designed to account for dynamic loads by increasing the design live loads by a semi-empirical “impact factor” or “dynamic load allowance”.

Recently, there have been many programs of research, discussing the effect of the characteristics of a bridge or a vehicle on the dynamic response of a bridge such as: the programs in U.S.A., U.K. and Canada [23], in the Organization for Economic Cooperation and Development (O.E.C.D.) in Switzerland [24] etc. Among the important studies on this field, one must especially refer to the important experimental research by Cantieri [25], dealing with different models of moving loads.

From the three factors (vehicle speed, matching of bridge-vehicle natural frequencies and irregularities and roughness of bridge deck-surface) which affect the vibration of a bridge, only the third one has been extensively studied in the last years, mainly by experimental methods.

It is well known and mentioned in many classical dynamics books such as Rogers [26], Timoshenko *et al* [27], that a concentrated mass attached to beam affects sometimes strongly its dynamic characteristics. On the other hand, there are many publications (mainly using a F.E.M. analysis), dealing more with the influence of the mass forces (mass inertia, centripetal, Coriolis forces etc) on the dynamic behavior of a bridge such as Szyszkowski & Sharbati [28] and Sharbati & Szyszkowski [29] and only a few dealing with these problems through a pure theoretical investigation such as Michaltsos *et al* [18], Michaltsos & Kounadis [30], and Reis & Pala [31].

A lot of semi-analytical approaches were recently applied for solving dynamic problems in engineering as for example Werme [32], Mamandi & Kargarnovin [33], Ju [34], Wang *et al* [35], Ghasemzadeh *et al* [36], while some researchers obtained interesting results using the decomposition method in engineering problems by Adomian [37, 38] which is an extension of the Ito integral, such as Hashim *et al* [39], Yahya *et al* [40], Duan & Rach [41], and Emad *et al* [42]. All the above mentioned publications ignore the alteration of the dynamic characteristics of the bridge during the vehicle passage.

This paper deals with the influence of a moving mass-load on the eigenfrequencies and the dynamic deformations of a beam, taking into account the aforementioned alteration of the eigenfrequencies due to the mass-load movement. A 2 DOF model is considered to study the dynamic behavior of the beam based on modal superposition and the continuum approaches. The gathered non-linear system of equations of motion is solved with the aid of a commercial symbolic manipulator. Characteristic examples are presented and useful conclusions are gathered.

2 ANALYTICAL FORMULATION

2.1 The mass M on the beam

Let us consider the beam of Figure 1 with mass per unit length m and a concentrated mass M at point A which is always in contact with the beam during its vibration. Thus, the rotational moment of inertia J_M does not affect the beam's vibrations.

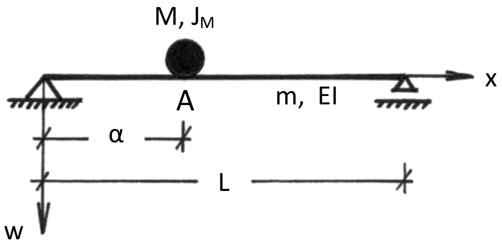


Figure 1. Beam with a mass M attached to point A

The equation of the free motion of the beam-mass system is given by:

$$EI\,w'''' + [m + M\cdot\delta(x-\alpha)]\,\ddot{w} = 0 \tag{1}$$

One can search for a solution in the form of separate variables:

$$w(x,t) = W(x)\cdot T(t) \tag{2}$$

Introducing Eq(2) into Eq(1), one gets:

$$EIW''''T + [m + M\cdot\delta(x-\alpha)]W\ddot{T} = 0$$

or

$$\frac{EIW''''}{[m + M\cdot\delta(x-\alpha)]W} = -\frac{\ddot{T}}{T} = \omega^2$$

and therefore, the equation giving the shape function W(x) of the beam is:

$$EIW'''' - \omega^2[m + M\cdot\delta(x-\alpha)]W = 0 \tag{3}$$

In order to apply the Galerkin's approach, we set:

$$W(x) = c_1X_1(x) + c_2X_2(x) + \cdots\cdots + c_nX_n(x) \tag{4}$$

where c_i are unknown coefficients to be determined and X_i are arbitrarily chosen functions of x, which satisfy the boundary conditions. As such functions one can choose the shape functions of the corresponding beam without the concentrated mass M.

Introducing Eq(4) into Eq(3), one obtains:

$$EI\sum_{\rho=1}^{n} c_\rho X_\rho'''' - \omega^2[m + M\cdot\delta(x-\alpha)]\sum_{\rho=1}^{n} c_\rho X_\rho = 0 \tag{5a}$$

Taking into account that $X_\rho(x)$ satisfy the equation of free motion:

$$X_\rho'''' - \frac{m\omega_\rho^2}{EI}\cdot X_\rho = 0 \tag{5b}$$

where ω_ρ the eigenfrequencies of the corresponding beam without the mass M, Eq(5a) becomes:

$$\left.\begin{aligned} &\sum_{\rho=1}^{n} c_\rho \lambda_\rho X_\rho - \lambda[1 + \mu\cdot\delta(x-\alpha)]\sum_{\rho=1}^{n} c_\rho X_\rho = 0 \\ &\text{where: } \lambda_\rho = \frac{m\omega_\rho^2}{EI}, \quad \lambda = \frac{m\omega^2}{EI}, \quad \mu = \frac{M}{m} \end{aligned}\right\} \tag{5c}$$

Multiplying the above by $X_k(x)$ and integrating the outcome from 0 to L, one obtains:

$$\left.\begin{aligned} &c_k\lambda_k\int_0^L X_k^2\,dx - \lambda\left[c_k\int_0^L X_k^2\,dx + \mu\sum_{\rho=1}^{n} c_\rho X_\rho(\alpha)\cdot X_k(\alpha)\right] = 0 \\ &\text{with} \quad k = 1 \text{ to } n \end{aligned}\right\} \tag{6a}$$

For a single span beam, one has:

$$\left.X_k = \sin\frac{k\pi x}{L}, \quad \lambda_k = \left(\frac{k\pi}{L}\right)^4\right\} \tag{6b}$$

Introducing Eq(6b) into Eq(6a) the following system is obtained:

$$\left.\begin{aligned} &c_1\left\{\frac{L}{2}(\lambda_1-\lambda) - \lambda\mu(\sin\frac{\pi\alpha}{L})^2\right\} - c_2\lambda\mu\sin\frac{2\pi\alpha}{L}\sin\frac{\pi\alpha}{L} - \cdots - c_k\lambda\mu\sin\frac{k\pi\alpha}{L}\sin\frac{\pi\alpha}{L} - \cdots - c_n\lambda\mu\sin\frac{n\pi\alpha}{L}\sin\frac{\pi\alpha}{L} = 0 \\ &-c_1\lambda\mu\sin\frac{\pi\alpha}{L}\sin\frac{2\pi\alpha}{L} + c_2\left\{\frac{L}{2}(\lambda_2-\lambda) - \lambda\mu(\sin\frac{2\pi\alpha}{L})^2\right\} - \cdots - c_k\lambda\mu\sin\frac{k\pi\alpha}{L}\sin\frac{2\pi\alpha}{L} - \cdots - c_n\lambda\mu\sin\frac{n\pi\alpha}{L}\sin\frac{2\pi\alpha}{L} = 0 \\ &\cdots\cdots\cdots \\ &-c_1\lambda\mu\sin\frac{\pi\alpha}{L}\sin\frac{k\pi\alpha}{L} - c_2\lambda\mu\sin\frac{2\pi\alpha}{L}\sin\frac{k\pi\alpha}{L} - \cdots + c_k\left\{\frac{L}{2}(\lambda_k-\lambda) - \lambda\mu(\sin\frac{k\pi\alpha}{L})^2\right\} - \cdots - c_n\lambda\mu\sin\frac{n\pi\alpha}{L}\sin\frac{k\pi\alpha}{L} = 0 \\ &\cdots\cdots\cdots \\ &-c_1\lambda\mu\sin\frac{\pi\alpha}{L}\sin\frac{n\pi\alpha}{L} - c_2\lambda\mu\sin\frac{2\pi\alpha}{L}\sin\frac{n\pi\alpha}{L} - \cdots - c_k\lambda\mu\sin\frac{k\pi\alpha}{L}\sin\frac{n\pi\alpha}{L} - \cdots + c_n\left\{\frac{L}{2}(\lambda_n-\lambda) - \lambda\mu(\sin\frac{n\pi\alpha}{L})^2\right\} = 0 \end{aligned}\right\}$$

(6c)

From the above system of Eqs(6c) one can determine:

$$\left.\begin{array}{l}
1)\ \text{Then values of } \lambda = \dfrac{m\omega^2}{EI} \text{ as a function of } \alpha:\ \lambda(\alpha) \\
2)\ \text{The constants } c_1, c_2, \cdots, c_n \text{ as functions of } \alpha:\ c_1(\alpha), c_2(\alpha), \cdots, c_n(\alpha) \text{ and thus} \\
3)\ \text{The deformation of the bridge}: W(x,\alpha) = \sum_{k=1}^{n} c_k(\alpha)\cdot X_k(x), \text{ with } X_k = \sin\dfrac{k\pi x}{L}
\end{array}\right\}$$

The unknown constants can be determined from Eqs(6c) as follows:

$$\left.\begin{array}{l}
c_1(\alpha) = c_1(\alpha) \\
c_2(\alpha) = c_1(\alpha)\dfrac{c_2(\alpha)}{c_1(\alpha)} \\
\dots\dots\dots\dots\dots\dots\dots \\
c_n(\alpha) = c_1(\alpha)\dfrac{c_n(\alpha)}{c_1(\alpha)}
\end{array}\right\} \tag{7a}$$

A simple numerical application with m=5 gives the following Table 1.

Table 1. Relations between constants for the first 5 eigenfrequencies and various beam lengths

L	15m	25m	50m	100m
c_2/c_1	(-0.02 to 0.02) c_1	(-0.012 to 0.0015) c_1	(-0.0075 to 0.0075) c_1	(-0.0045 to 0.0045) c_1
c_3/c_1	(-0.006 to 0.004) c_1	(-0.0035 to 0.0025) c_1	(-0.0025 to 0.0017) c_1	(-0.0013 to 0.0007) c_1
c_4/c_1	($-2*10^{-4}$ to 10^{-4}) c_1	~0	~0	~0

The above Table 1 verifies that the first is the predominant mode, while the rest modes have an influence less than 2% (for spans up to 15m) and 7 to $4^0/_{00}$ (for bigger spans).

Therefore, Eq(4) can be rewritten with satisfactory accuracy as follows:

$$w(x) \cong c_1(\alpha)\cdot X_1(x) \tag{7b}$$

where $c_1(\alpha)$ is a coefficient the form of which must be determined.

The following equation is valid:

$$w(x,\alpha) = \sum_n c_n(\alpha)\cdot X_n(x) = w_o(x,\alpha) \tag{7c}$$

where for the loading shown in Figure 1, one can write:

$$\left.\begin{array}{ll}
w_o(x,\alpha) = \dfrac{Mg(L-\alpha)}{6EI}\cdot\dfrac{x}{L}\cdot[L^2-(L-\alpha)^2-x^2] & \text{for } 0\le x\le\alpha \\
w_o(x,\alpha) = \dfrac{Mg\alpha}{6EI}\cdot\dfrac{L-x}{L}\cdot[L^2-\alpha^2-(L-x)^2] & \text{for } \alpha\le x\le L
\end{array}\right\} \tag{7d}$$

From Eqs(7b) and (7c), one can determine the expression of $c_1(\alpha)$ as follows:

$$c_1(\alpha) = \frac{2}{L} \cdot \int_0^L w_o(x,\alpha) X_1(x)\,dx = \frac{2MgL^3}{\pi^4 EI} \cdot \sin\frac{\alpha\pi}{L} = c_1 \cdot \sin\frac{\alpha\pi}{L} \tag{7e}$$

where c_1 is an arbitrary constant that can set equal to 1.

2.2 The moving mass M on the beam

For a mass-load M, moving on the beam of Figure 2, the equation of motion is:

$$EI w'''' + m\ddot{w} = M(g - \ddot{w}) \cdot \delta(x - \alpha) \tag{8a}$$

Obviously, Eq(8a) is not complete since it does not contain all mass inertia forces. The complete equation has been presented by Michaltsos & Kounadis [30], where the influence of the Centrifugal and Coriolis forces has been studied, in relation to the ratio M/mL and the velocity υ of the moving mass. It has been shown that including these forces in the analysis has similar effect on the dynamic response of the beam as the moving mass-load consideration.

The influence of the above mentioned terms varies from 1.5 to 5% for beams with normal characteristics, and from 5 to 20% for beams with characteristics not used in the practice.

There are also the terms of the Mass’ rotational moment of inertia and of the wheelbase of a real vehicle that affect the motion of a beam and which have been presented by Michaltsos [43].

The complete form of the equation of motion containing in addition to the above terms and several others such as the tangential velocity component due to the deformation of the beam, is given in paper and the book by Michaltsos & Raftoyiannis [44, 45]. Therefore, the completeness of an equation is something relevant, depending each time on the purpose it serves.

However, this paper does not aim to study the accuracy or correctness of this equation, but the efficiency of the solution. Undoubtedly, similar results can be gathered from the study of the corresponding complete equation. We are searching for a solution in the form:

$$w(x,t) = \sum_{\rho=1}^{n} W_\rho(x,\alpha)\Phi_\rho(t) \tag{8b}$$

where $\Phi_\rho(t)$ are the unknown time functions (under determination) and $W_\rho(x,\alpha)$ are given by Eq(4) with $c_\rho = c_\rho(\alpha)$, as they have been obtained from the solution of Eqs(6c).

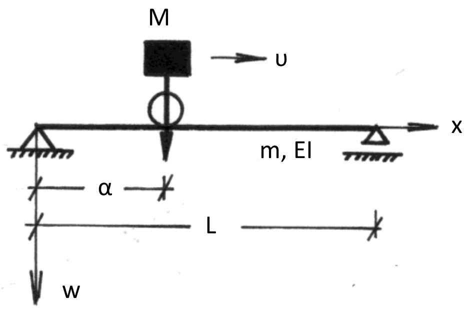


Figure 2. Beam with a mass M moving with velocity υ

From Eq(8b) one obtains:

$$\ddot{w}(x,t)=\sum_{\rho=1}^{n}W_\rho(x,\alpha)\ddot{\Phi}_\rho(t)+2\sum_{\rho=1}^{n}\dot{W}_\rho(x,\alpha)\dot{\Phi}_\rho(t)+\sum_{\rho=1}^{n}\ddot{W}_\rho(x,\alpha)\Phi_\rho(t) \qquad (8c)$$

in which the distance α is a relation of t.
Because of Eqs(8b) and (8c) , Eq(8a) can be written as follows:

$$E\,I\sum_{\rho=1}^{n}W_\rho^{''''}(x,\alpha)\ddot{\Phi}_\rho(t)+[m+M\delta(x-\alpha)]\sum_{\rho=1}^{n}(W_\rho\ddot{\Phi}_\rho+2\dot{W}_\rho\dot{\Phi}_\rho+\ddot{W}_\rho\Phi_\rho)=Mg\delta(x-\alpha) \qquad (8d)$$

Given that Eq(3) is valid, the above Eq(8d) becomes:

$$\left.\begin{array}{l}[m+M\,\delta(x-\alpha)]\sum_{\rho=1}^{n}\{W_\rho(x,\alpha)\ddot{\Phi}_\rho(t)+2\dot{W}_\rho(x,\alpha)\dot{\Phi}(t)+[\ddot{W}_\rho(x,\alpha)+\omega_\rho^2 W_\rho(x,\alpha)]\,\Phi_\rho(t)\}= \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=Mg\delta(x-\alpha) \\ \text{where: }\ \omega_\rho^2=\dfrac{E\,I}{m}\cdot\lambda_\rho(\alpha)\quad\text{and}\quad\lambda_\rho\quad\text{from Eq6c)}\end{array}\right\} \qquad (8e)$$

Multiplying successively the above equation by $X_1, X_2, \cdots, X_n$, integrating the outcome from 0 to L, taking into account Eq(4) and because of the orthogonality condition of the shape function $X_\rho(x)$, one concludes to the following differential system:

$$\left.\begin{array}{l}(a_{11}\ddot{\Phi}_1+2b_{11}\dot{\Phi}_1+\gamma_{11}\Phi_1)+\cdots\cdots+(a_{1n}\ddot{\Phi}_n+2b_{1n}\dot{\Phi}_n+\gamma_{1n}\Phi_n)=\mu g X_1(\alpha) \\ (a_{21}\ddot{\Phi}_1+2b_{21}\dot{\Phi}_1+\gamma_{21}\Phi_1)+\cdots\cdots+(a_{2n}\ddot{\Phi}_n+2b_{2n}\dot{\Phi}_n+\gamma_{2n}\Phi_n)=\mu g X_2(\alpha) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (a_{k1}\ddot{\Phi}_1+2b_{k1}\dot{\Phi}_1+\gamma_{k1}\Phi_1)+\cdots\cdots+(a_{kn}\ddot{\Phi}_n+2b_{kn}\dot{\Phi}_n+\gamma_{kn}\Phi_n)=\mu g X_k(\alpha) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (a_{n1}\ddot{\Phi}_1+2b_{n1}\dot{\Phi}_1+\gamma_{n1}\Phi_1)+\cdots\cdots+(a_{nn}\ddot{\Phi}_n+2b_{nn}\dot{\Phi}_n+\gamma_{nn}\Phi_n)=\mu g X_n(\alpha)\end{array}\right\} \qquad (9a)$$

where:

$$\left.\begin{aligned}
a_{k\rho} &= \frac{L}{2} c_{k\rho}(\alpha) + \mu \sum_{\zeta=1}^{n} c_{\zeta\rho}(\alpha) X_{\zeta}(\alpha) X_{k}(\alpha) \\
b_{k\rho} &= \frac{L}{2} \dot{c}_{k\rho}(\alpha) + \mu \sum_{\zeta=1}^{n} \dot{c}_{\zeta\rho}(\alpha) X_{\zeta}(\alpha) X_{k}(\alpha) \\
\gamma_{k\rho} &= \frac{L}{2} [\ddot{c}_{k\rho}(\alpha) + \omega_{\rho}^{2} c_{k\rho}(\alpha)] + \mu \sum_{\zeta=1}^{n} (\ddot{c}_{k\rho}(\alpha) + \omega_{\rho}^{2} c_{k\rho}(\alpha)) X_{\zeta}(\alpha) X_{k}(\alpha) \\
&\text{with: } \alpha = \upsilon \cdot t, \text{ and } \mu = M/m
\end{aligned}\right\} \quad (9b)$$

The above differential system of $\Phi_1(t), \Phi_2(t), \cdots, \Phi_n(t)$, can be solved only numerically.

2.3 The eigenfrequencies

It is obvious that the term $\omega_k^2(t)$ is a very complicated function. This fact makes the solution of the above non-linear differential system particularly difficult and sometimes impossible. A simple and efficient way for the solution of such a system is to simulate the term $\omega_k^2(t)$ with an expression having a form easy to manage. Plotting the functions of $\lambda_k(\alpha)$ determined by Eq(6c), one obtains diagrams like the ones of Figure 3.

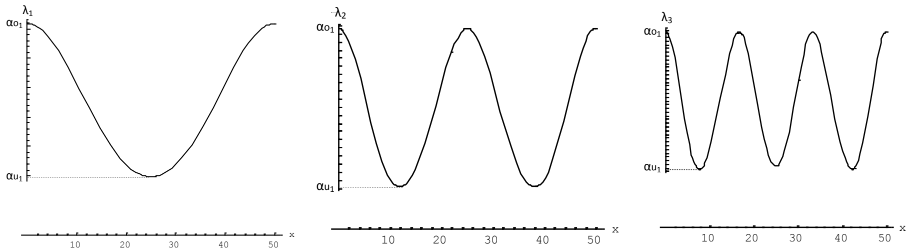


Figure 3. The form of the three first eigenfrequencies due of a moving mass M

As it can be ascertained from the studied examples, the following function expresses the real form of λ_k with satisfactory accuracy:

$$\left.\begin{aligned}
\lambda_k &= A_k + B_k \cdot \cos\frac{2 \cdot k \cdot \pi \cdot \alpha}{L} \\
\text{with: } A_k &= \frac{\alpha_{ok} + \alpha_{uk}}{2}, \quad B_k = \frac{\alpha_{ok} - \alpha_{uk}}{2}
\end{aligned}\right\} \quad (10)$$

3 NUMERICAL RESULTS AND DISCUSSION

The main goal of this paper is to study the influence of a moving mass on the eigenfrequencies of a beam and therefore to investigate the accuracy of the up to now established and applied analyses in the design.

Let us consider a single-span bridge with length L=50m, moment of inertia $I=0.30m^4$ and mass per unit length m=1000 kg/m. The bridge is made from homogeneous and isotropic material with modulus of elasticity $E = 2.1 \cdot 10^{10}\ dN/m^2$.

The bridge is subjected to a vehicle of mass $M = \mu \cdot m$ (with μ = 1, 5, 10, 20), moving with constant velocity υ (υ = 10, 20, 30, 40 m/sec). Without restriction of the generality, the aforementioned vehicle can be considered as a concentrated load-mass. Applying the expressions derived in §2 one obtains the following plots shown in Figures 4, 5 and 6. Note that the last value (μ=20) is not realistic and it is added for purely theoretical purpose.

a) The plots of Figure 4 show the variation of the first eigenfrequency due to a moving mass M with μ= 1, 5, 10 and 20.

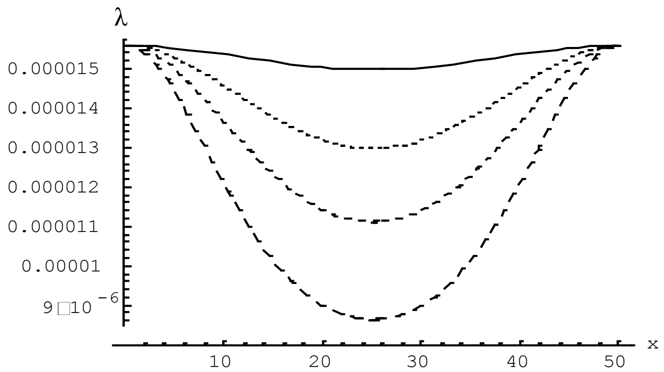


Figure 4. Variation of the first eigenfrequency $\lambda_1 = m\omega_1^2/(EI)$ in relation to the position of a moving mass, for μ=1 (__), μ=5 (….) , μ=10 (- - -) and μ=20 (_ _)

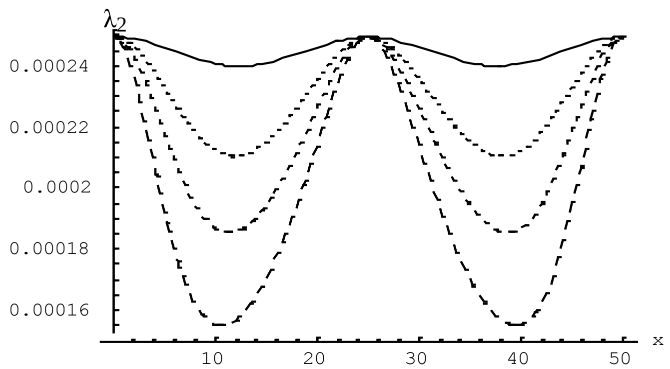


Figure 5. Variation of the second eigenfrequency $\lambda_2 = m\omega_2^2/(EI)$ in relation to the position of a moving mass, for μ=1 (__), μ=5 (….) , μ=10 (- - -) and μ=20 (_ _)

One ascertains that λ_1 decreases (when the mass approaches the middle of the bridge span) almost ~4% for μ=1, ~15% for μ=5, ~28% for μ=10 and ~40% for μ=20.

b) The plots of Figure 5 show the variation of the second eigenfrequency due to a moving mass M with μ= 1, 5, 10 and 20. One ascertains that λ_2 decreases almost ~4% for μ=1, ~16% for μ=5, ~25% for μ=10 and ~38% for μ=20.

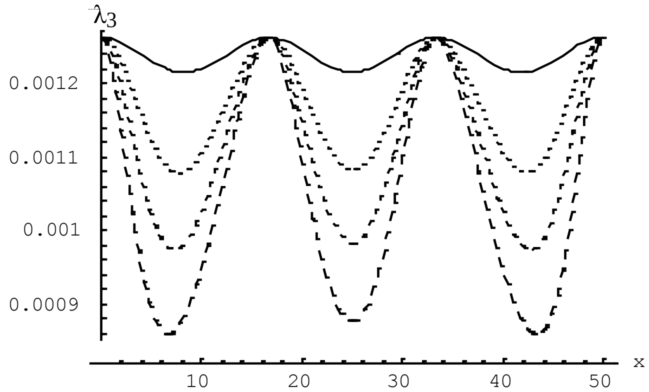


Figure 6. Variation of the third eigenfrequency $\lambda_3 = m\omega_3^2/(EI)$ in relation to the position of a moving mass, for μ=1 (__), μ=5 (….) , μ=10 (- - -) and μ=20 (_ _)

c) The plots of Figure 6 show the variation of the third eigenfrequency due to a moving mass M with μ= 1, 5, 10 and 20. In this case, λ_3 decreases almost ~3% for μ=1, ~14% for μ=5, ~23% for μ=10 and ~35% for μ=20.

3.1 Eigenfrequencies simulation

Applying the procedure presented in §2, one can determine the factors A_{1k} and A_{2k}, shown in the following Table 2, in order to simulate the eigenfrequencies according to Eqs(10).

Table 2. Factors A_{1k} and A_{2k} for various values of μ

	A_{11}	A_{21}	A_{12}	A_{22}	A_{13}	A_{23}
μ=1	0.00001558	0.00001498	0.0002474	0.0002399	0.001262	0.001215
μ=5	0.00001558	0.0000131	0.0002474	0.0002110	0.001262	0.001080
μ=10	0.00001558	0.0000112	0.0002474	0.0001865	0.001262	0.000957

In the following Figure 7, one can see the simulation of the first three eigenfrequencies of the bridge for μ=5.

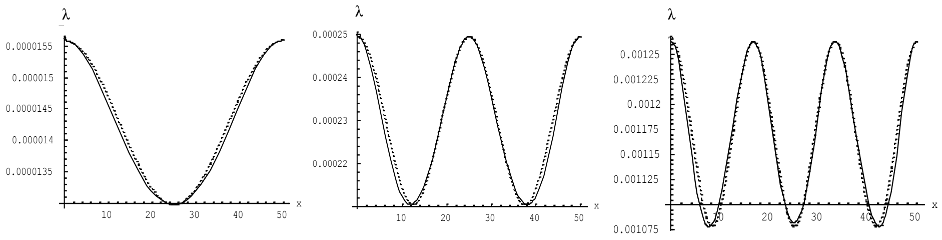


Figure 7. Simulation of the first three eigenfrequencies

3.2 The motion of the bridge

Solving the differential system of Eqs(9a) with the aid of a commercial symbolic manipulator, i.e., Mathematica package [46], one obtains the following plots of Figures 8, 9 and 10.

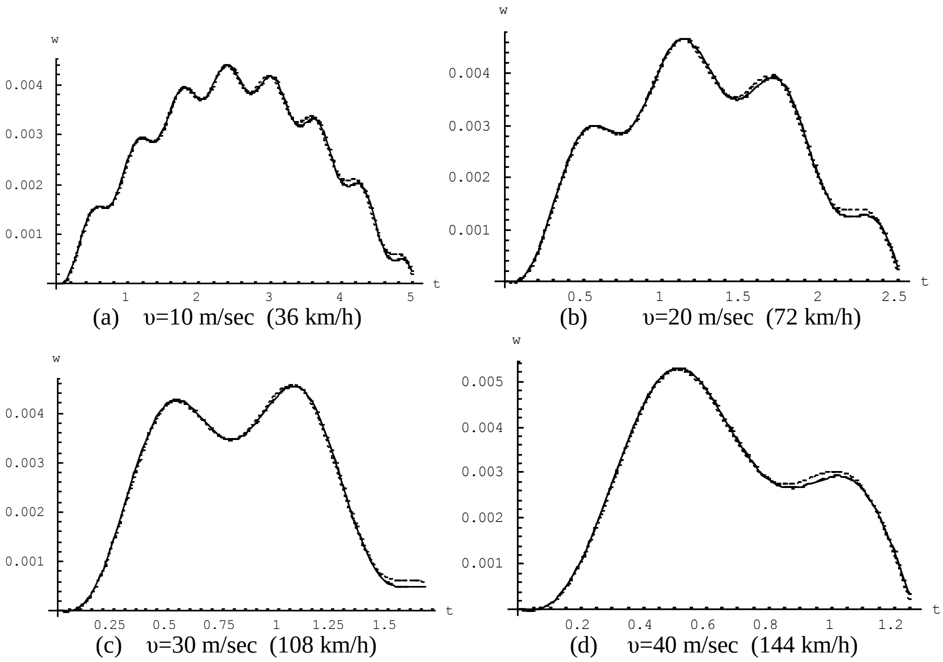


Figure 8. The oscillations of the mid-point of the bridge for μ=1 and various values of υ
(….) Load F=Mg, (- - -) Influence of the inertia of a mass M, (__) Interaction Beam-mass M

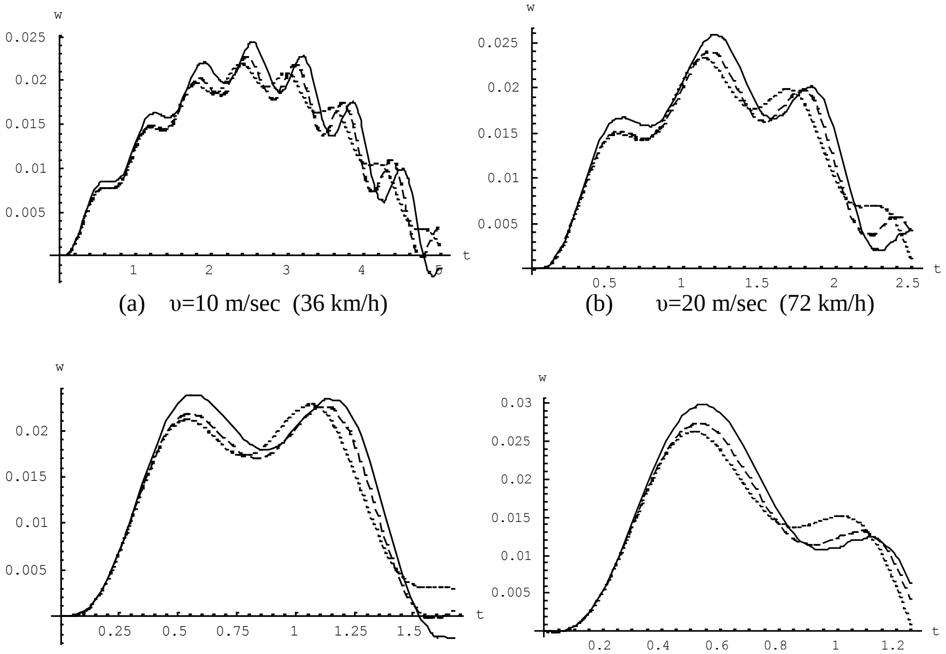
In the above plots, only the moving masses with $\mu = 1$, 5 and 10 are presented, since masses with $\mu > 10$ are considered as non-realistic cases. Each plot contains three cases:

- The motion of the middle of the bridge span due to a moving load F=Mg, ignoring the influence of the mass inertia.
- The motion of the middle of the bridge span due to a moving load-mass M,

where the inertia of the mass M is included, and finally

- The motion of the middle of the bridge span due to a moving load-mass M taking into account the interaction between the beam and the mass M.

Figure 8 shows the motion of the mid-point of the bridge for μ = 1 and various velocities. In this case, the differences are negligible and the deformations amount, for case (b), up to ~1% more than the ones of case (a), and also ~1% for case (c) more than the ones of case b.



(a) υ=10 m/sec (36 km/h)

(b) υ=20 m/sec (72 km/h)

(c) υ=30 m/sec (108 km/h)

(d) υ=40 m/sec (144 km/h)

Figure 9. The oscillations of the mid-point of the bridge for μ=5, and various values of υ (….) Load F=Mg, (- - -) Influence of the inertia of a mass M, (__) Interaction Beam-mass M

Figure 9 shows the motion of the mid-point of the bridge for μ = 5 and various velocities. In this case and for low velocities, the differences of the deformations amount, for case (b), up to ~5% more than the ones of case (a), and furthermore ~7% for case (c), more than the ones of case (b), while for higher speeds the corresponding differences are ~6% and ~9% .

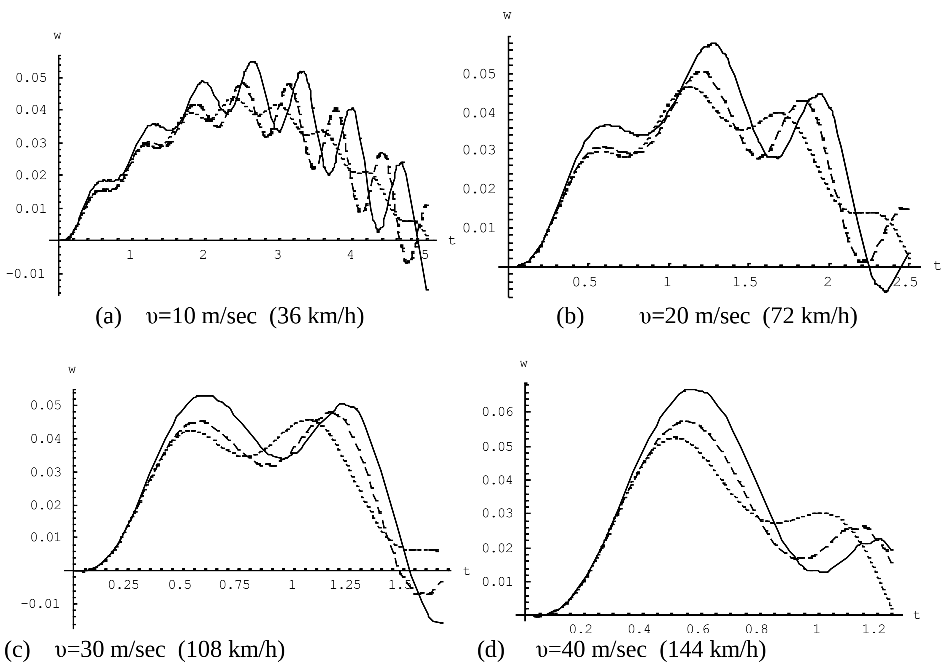


Figure 10. The oscillations of the mid-point of the bridge for μ=10 and various values of υ (….) Load F=Mg, (- - -) Influence of the inertia of a mass M, (__) Interaction Beam-mass M

Figure 10, shows the motion of the mid-point of the bridge for $\mu = 10$ and various velocities.

In this case and low speeds the differences of the deformations amount, for case (b), up to ~10% more than the ones of the case (a) and also ~12% for case (c), more than the ones of case (b), while for higher speeds the corresponding differences are ~10% and ~18% .

4 CONCLUSIONS

Based on the analytical models and the numerical examples presented herein, one can draw the following conclusions:

- The non-linear differential equation governing the vertical motion of a bridge under the action of a moving mass-load is formulated and solved through the use of the simulation procedure, and the aid of a commercial symbolic manipulator, while it was investigated through relative examples.
- The motion of a mass-load M affects strongly the eigenfrequencies of the bridge, especially when the mass M moves near the mid-point of the span. The eigenfrequencies decrease from ~4% for masses with $\mu = 1$, and up to ~25% for masses with $\mu = 10$, while for the non-realistic case of a mass with $\mu = 20$ the decrease is in the order of ~35 to ~40%, depending on the velocity

of the vehicle.

- Regarding the influence of the inertia of the mass (without interaction between beam and mass), the eigenfrequencies decrease from ~1% for μ =1 up to ~10% for μ =10. The influence of the mass on the motion of the beam (due to the beam-mass interaction that occurs through the eigenfrequencies' alteration) is negligible for μ =1 (~0.5 to 1.5%) while it becomes considerable for μ =5 (~7 to 9%) and significant for μ =10 (~12 to 18%), in comparison to the ones found if only the inertia of the mass is taken into account.
- The above theoretical findings show that for cases usually met in practice, the results from an analysis considering only the load F=Mg (i.e. without the mass inertia forces) must be increased from a minimum 2.5% to a maximum 20%, depending on the ratio μ = M/m.
- This investigation is limited to bridges with maximum span-length 80 to 100 meters. The length of the bridge is an important factor involved in the computations through the ratio M/mL, but it is not studied herein.

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