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THE INFLUENCE OF PRESTRESSING ON THE Stability of the main beams of a low-passage bridge

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ABSTRACT: The phenomenon of twisting appears in bending beams as a special case of the lateral-torsional buckling of a beam. In this paper, we will study the influence of the prestressing on the stability (twisting stability) of a simply supported beam. The case of the prestressing by rectilinear tendons is studied and numerical applications are presented.

KEYWORDS: Prestressing, Stability, Low-Passage Bridge

1 INTRODUCTION

The origin of the prestressed steel members dates back many years ago. This technique is attributed to Paxton, who in 1851 used prestressed steel beams for the building of Crystal Palace. In 1907, Koenen was the first which proposed prestressed steel bars, many years before the application of prestressing in concrete [1].

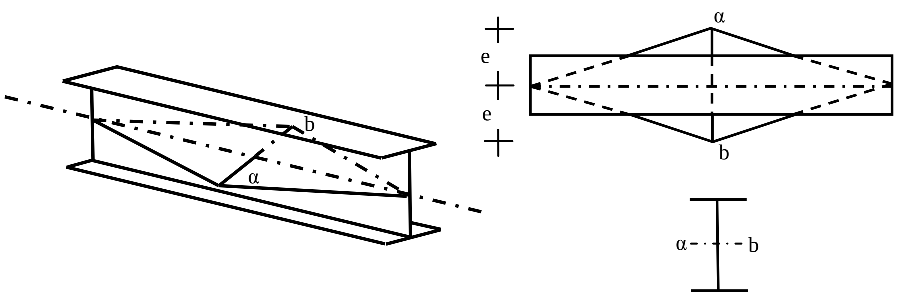


Figure 1. Prestressing System of a Steel Beam

The father of prestressing, mainly in concrete, is Eugène Freyssinet, who in 1928 defined prestressing as a technique which consists in increasing the capacity of a material to undertake greater loads than the ones of a material without prestress.
Comparing the steel prestressed members to the concrete prestressed ones, we have to indicate the following:

a) The concrete prestressed members are friables and are underlain to the shrinking phenomenon. The above, are unknown to a steel member.
b) This technique further raises both the quality and the resistance to tension and compression characteristics of the steel. In opposite, the resistance to tension in reinforced concrete is, in fact, negligible or nonexistent.

The prestressing of members is easily applicable both on new and existing structures and especially for the strengthening of existing bridges [2].
As it is generally known, while an installation of prestressing tendons doubles the load-carrying capacity of a structure, it actually increases also the load carrying capacity if buckling of the structure is considered [3,4].
The research on this last field of instability, is rather poor, and the existing publications examine this problem mainly through experimental way [5,6].
The instability of a beam, may be appeared not only as the classical buckling of a column, but also as the twisting phenomenon (a special case of the lateral-torsional buckling).
This phenomenon, appears in bending beams with or without joins (or obligations) along the beam-span.
As an example of the first case we would mention the simply supported beam, while as an example of the second case we would mention the main-beams of a bridge with deck of under-passage. Let us see the system of figure 1, which is a back pushing system. We consider in addition tha the rods or cables "a" and "b" are not loaded. By applying an axial force P, we will have evidently a critical load bigger than the critical buckling one. A prestressing cable operates like the above system in its limit case (with e = 0). Supposing that the ends of the beam are unmoved a cable reacts and acts like a back – pushing system just when an eccentricity "e" appears at the beginning of the twisting phenomenon of the beam.
In this paper we will study the increasing of the stability (twisting-stability) of a simply supported beam prestressed by rectilinear tendons, that is the usual way of prestressing in steel beams. Without restriction of the generality, the exposed method can be also applied in other forms of tendons like the one of Figure 1b (that is rectilinear by parts).

2 INTRODUCTORY CONCEPTS

1. The beams in use, have usually the moment of inertia I_y much bigger than the I_z one. In the case of a beam like the one of figure 1a, where $I_z << I_y$, appears the phenomenon of twisting (a special case of lateral-torsional buckling).

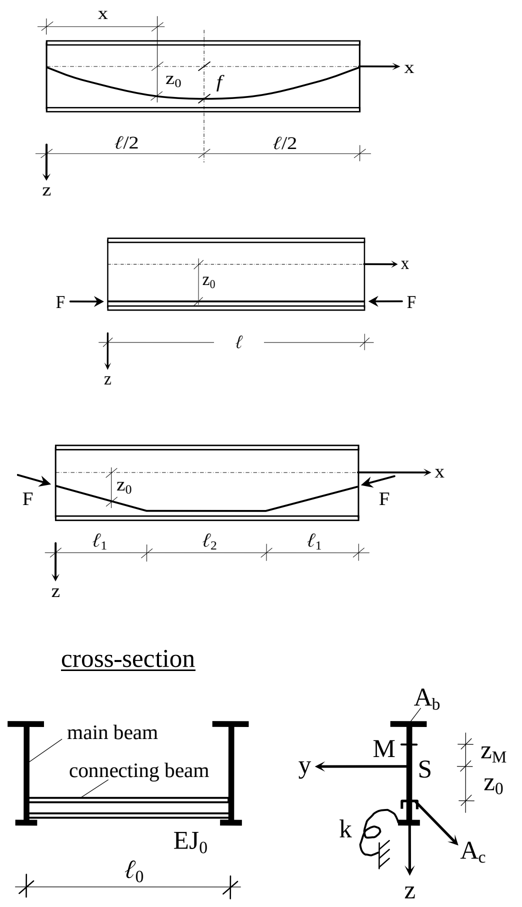


Figure 2. Prestressed Steel Beam Cable Arrangements

2. Let us consider now the beam of Fig. 2, which is prestressed by cables of random form, given by the equation:

$$z_o = e_z + z(x) \tag{1}$$

Where we suppose, in addition, that the points P of anchorage of the cables at the edges of the beam are located at a distance e_z from the gravity center S of the beam's cross-section.(see Fig. 4).

3. We assume that $w << \upsilon$ and thus the terms due to w can be neglected.
4. Finally the beam may be restrained against torsion, as a member, for example, of a bridge, by a spring of constant k (see Fig.2).
5. The external loads produce the moment $M_y(x)$, which after the deformation of the beam is analyzed to (see Fig. 3):

$$\left.\begin{aligned} M_{\tilde{y}} &= M_y \cos\varphi \cong M_y \\ M_{\tilde{z}} &= -M_y \sin\varphi \cong -M_y \cdot \varphi \end{aligned}\right\} \quad (2)$$

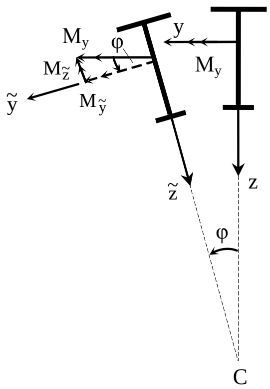


Figure 3. Analysis after the deformation of the Moment produced by the external loads

3 ANALYSIS

We will proceed using the method of the developed potential energy. The produced work is due to the external forces and to the internal ones.

3.1. The work of the internal forces.

Taking into account the cross-section's warping, the work of the internal forces is:

$$E_i = \frac{1}{2}\int_0^{\ell}(E\, I_z \upsilon''^2 + E\, I_\omega \varphi''^2 + G\, I_d \varphi'^2)dx \quad (3)$$

3.2. The work of the external forces.

3.2.1. The work of the force F.

Because of F (the force of prestressing), the developed tension at a random point $B(y_B, z_B)$ is equal to $\sigma_B = -F_x\left(\frac{1}{A_b} + \frac{z_B}{I_y} e_z\right)$, while the length of the corresponding fibre is equal to $\frac{1}{2}\int_0^{\ell}(\upsilon_B'^2 + w_B'^2)dx$. Therefore the produced work will be:

$$E_F = -\frac{F_x}{2}\int_0^{\ell}\oint_{A_b}\left(\frac{1}{A_b}+\frac{z_B}{I_y}e_z\right)(\upsilon_B'^{\,2}+w_B'^{\,2})dA_b dx \qquad (4)$$

The displacement υ_B of the point b, because of the rotation φ, is related to the displacement υ of the gravity center by the following relation:

$$\upsilon_B = \upsilon - \overset{*}{z}_B\,\varphi \qquad (a)$$

On the other hand it is valid:

$$z_B = z_M + \overset{*}{z}_B \qquad (b)$$

where $\overset{*}{z}$ is shown in Fig. 4.
From the above (a) and (b) we get finally:

$$\left.\begin{aligned} \upsilon_B &= \upsilon - (z_B - z_M)\varphi \\ w_B &= y_B\varphi \end{aligned}\right\} \qquad (c)$$

From the first of the above we get:

$$\upsilon_B'^{\,2} = \upsilon'^2 + (z_B^2 + z_M^2 - 2z_M z_B)\varphi'^2 - 2\upsilon'\varphi'(z_B - z_M) \qquad (d)$$

Given that:

$$\left.\begin{aligned} &\int_{A_b} y_B\, dA_b = \int_{A_b} z_B\, dA_b = \int_{A_b} y_B z_B\, dA_b = 0 \\ &\int_{A_b} y_B^2\, dA_b = I_z \quad , \quad \int_{A_b} z_B^2\, dA_b = I_y \end{aligned}\right\} \qquad (e)$$

Equation (4), because of the above gives:

$$\left.\begin{aligned} &E_F = -\frac{F_x}{2}\int_0^{\ell}(\upsilon'^2 + i_M^2\varphi'^2 + 2z_M\upsilon'\varphi')\,dx - \frac{F_x}{2A_b}\int_0^{\ell}\left[(V_y - 2z_M I_y)\varphi'^2 - 2I_y\upsilon'\varphi'\right]\frac{e_z}{I_y}\,dx \\ &where: \quad V_y = \int_{A_b} z_B(y_B^2 + z_B^2)dA_b \quad , \quad i_M^2 = i_p^2 + z_M^2 \quad , \quad i_p^2 = \frac{I_p}{A_b} = \frac{I_y + I_z}{A_b} \end{aligned}\right\}$$

(5)

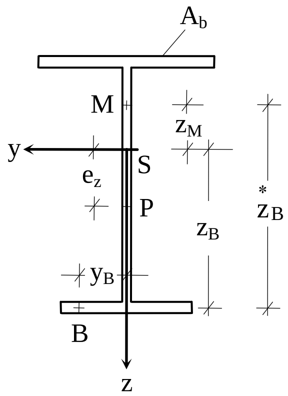


Figure 4. Position of random point B in relation to the Center of Gravity S

3.2.2. The work of $M_{\bar{z}}$.

If ψ is the angle between the axis x and the bended axis of the beam, will be $\psi = \upsilon'$ and thus $d\psi = \upsilon'' dx$ or $dE_M = -M_{\bar{z}}\upsilon'' dx$ and finally, because of Eqn. 2:

$$E_M = -\int_0^{\ell} M_{\bar{z}}\upsilon''\, dx = \int_0^{\ell} M_y \upsilon''\varphi\, dx \tag{6}$$

3.3. The produced work by the cables.

In Fig. 5, it is shown the deformed state of a cable of random form. We point out, that in the deformed position of the beam, the cable in addition to the pressure q_z acts also the pressure q_y, which reacts to the beam's displacement.
Keeping in mind that w_c is very small compared to υ_c (where w_c, υ_c are the displacements of the cable), we have that υ_c is connected to the displacement of S by the relation:

$$\upsilon_c = \upsilon - z_o\varphi \tag{7}$$

Projecting the cable on the plane (oxz) we have:

$$\left.\begin{aligned} F_x &= cons\tan t \\ F_z &= F_x z_o' \\ d\,F_z &= -q_z\, dx \end{aligned}\right\} \tag{8a}$$

And finally:

$$q_y = -F_x\, z_o'' \tag{8b}$$

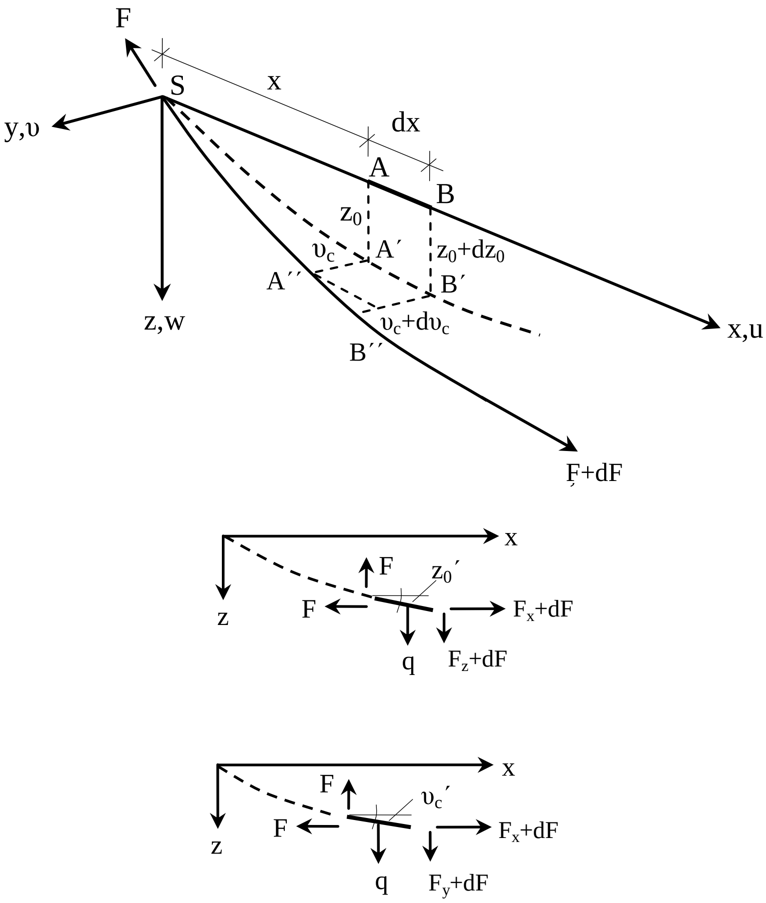


Figure 5. The deformed state of a cable of random form

Similarly, projecting the cable on the plane (oyz) we have (see Fig.5):

$$q_y = -F_x \upsilon_c'' \tag{8c}$$

3.3.1. Work produced by the forces q_y, q_z.

The produced work by q_y and q_z, taking into account that $w_c << \upsilon_c$ and equs (8), will be: $E_{F1} = \int_0^{\ell} (q_y \upsilon_c + q_z w_c)\, dx \cong -F_x \int_0^{\ell} \upsilon_c \upsilon_c'' dx$, or because of equation (7) we get:

$$E_{F1} = -F_x \int_0^{\ell} [\upsilon\upsilon'' - z_o\varphi\upsilon'' - \upsilon(z_o\varphi)'' + z_o\varphi(z_o\varphi)'']\,dx \qquad (9)$$

3.3.2. Work produced by the cable it-self.
From figure 5 we have:

- Coordinates of A'': (x, υ_c, zo)
- Coordinates of B'': (x+dx, $\upsilon_c + d\upsilon_c$, z_o+dz_o)

Therefore the length $A''B''$ will be:

$$A''B'' = \sqrt{dx^2 + d\upsilon_c{}^2 + dz_o{}^2} = dx\,\sqrt{1 + \upsilon_c'^2 + z_o'^2} = dx\left(1 + \frac{\upsilon_c'^2}{2} + \frac{z_o'^2}{2}\right)$$

Thus, the elementary part ds will have its final length equal to:

$$ds = \left(1 + \frac{z_o'^2}{2} + \frac{\upsilon_c'^2}{2}\right)dx$$

and the total length of the cable will be:

$$s = \int_0^{\ell}\left(1 + \frac{z_o'^2}{2} + \frac{\upsilon_c'^2}{2}\right)dx \qquad (e)$$

On the other hand, we know that the total elongation of a cable of length s and of area of cross-section A_c is:

$$\Delta s = \frac{F}{A_c E}\cdot s \qquad (f)$$

Therefore the produced work will be:

$$E_{F2} = F\cdot\Delta s = \frac{F^2}{A_c E}\int_0^{\ell}\left(1 + \frac{z_o'^2}{2} + \frac{\upsilon_c'^2}{2}\right)dx \qquad (g)$$

Or because of equ. (7), the above equ (g) becomes:

$$E_{F2} = \frac{F^2}{E\,A_c}\int_0^{\ell}\left(1 + \frac{z_o'^2}{2} + \frac{(\upsilon - z_o\varphi)'^2}{2}\right)dx \qquad (10)$$

3.4. The work of the spring.

We accept that M=kφ, and therefore: $d\ E_s = \frac{1}{2}\varphi\ M\ dx = \frac{1}{2}\,k\,\varphi^2\ dx$

or finally:

$$E_s = \frac{1}{2}\int_0^{\ell} k\,\varphi^2\,dx \tag{11a}$$

To determine the constant k, we consider that we have n beams along the main ones, which connect them, having moment of inertia I_o.

We consider in addition that the above beams of length ℓ_o are connected to the main-beams through semi-rigid connections of coefficient r.

Thus will be: $M_1 = \frac{n}{\ell} \cdot r \cdot \frac{4 E I_o}{\ell_o} \cdot \varphi = M = k\,\varphi$, and therefore:

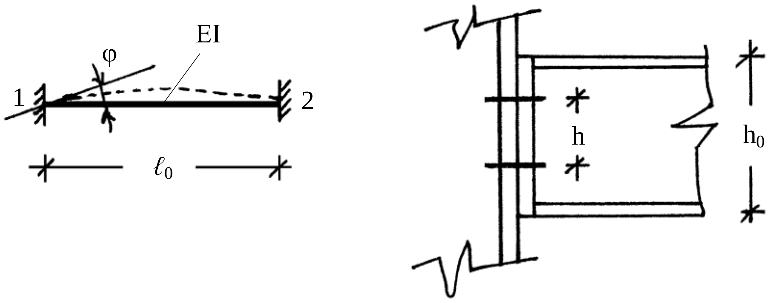


Figure 6. Form of the Semi-rigid connection

$$k = r \cdot \frac{4 n E I_o}{\ell \cdot \ell_o} \tag{11b}$$

Finally, the coefficient r, for a connection like the one in figure 6b will be:

$$r = \frac{\pi d^2 h h_o}{2 I_o} \tag{11c}$$

Where d is the diameter of the screw kernel.

3.5. The total work.

According to the previous analysis the total produced work is:

$$E = E_i + E_F + E_M + E_{F1} + E_{F2} + E_s \tag{12}$$

The following condition must be fulfilled:

$$\delta E = 0 \tag{13a}$$

Which concludes to the following equation:

$$
\left.
\begin{aligned}
\delta E = & \int_0^{\ell} (EI_z \upsilon'' \delta\upsilon'' + EI_{\omega} \varphi'' \delta\varphi'' + GI_d \varphi' \delta\varphi')\, dx \\
& - F_x \int_0^{\ell} (\upsilon' \delta\upsilon' + i_M^2 \varphi' \delta\varphi' + z_M \upsilon' \delta\varphi' + z_M \varphi' \delta\upsilon')\, dx \\
& - \frac{F_x e_z}{A_b I_y} \int_0^{\ell} \left[(V_y - 2z_M I_y) \varphi' \delta\varphi' - I_y \upsilon' \delta\varphi' - I_y \varphi' \delta\upsilon' \right] dx \\
& + \int_0^{\ell} M_y (\upsilon'' \delta\varphi + \varphi \delta\upsilon'')\, dx \\
& - F_x \int_0^{\ell} \begin{bmatrix} \upsilon \delta\upsilon'' + \upsilon'' \delta\upsilon - z_o \varphi \delta\upsilon'' - z_o \upsilon'' \delta\varphi - \upsilon \delta (z_o \varphi)'' - (z_o \varphi)'' \delta\upsilon \\ + z_o \varphi \delta (z_o \varphi)'' + z_o (z_o \varphi)'' \delta\varphi \end{bmatrix} dx \\
& + \frac{F^2}{E A_c} \int_0^{\ell} (\upsilon - z_o \varphi)' \delta (\upsilon - z_o \varphi)'\, dx + \int_0^{\ell} k\, \varphi\, \delta\varphi\, dx = 0
\end{aligned}
\right\} \quad (13b)
$$

After partial integration and some manipulation of Eqn. (14), we get integrated terms and integrals. The first are the boundary conditions while the last give the following differential equations of the problem:

$$
\left.
\begin{aligned}
& E\, I_z \upsilon'''' - (F\, C_o - \frac{F^2}{EA_c}) \upsilon'' + FC_o (z_M - \frac{e_z}{A_b}) \varphi'' + (2FC_o + \frac{F^2}{EA_c})(z_o \varphi)'' + (M_y \varphi)'' = 0 \\
& E\, I_{\omega} \varphi'''' - (G\, I_d - i_M^2 FC_o) \varphi'' + (z_M FC_o + 2z_o FC_o - \frac{e_z FC_o}{A_b} + \frac{z_o F^2}{EA_c}) \upsilon'' \\
& \qquad + \frac{e_z FC_o}{A_b I_y} (V_y - 2z_M I_y) \varphi'' + \frac{e_z FC_o}{A_b I_y} (V_y - 2z_M I_y) \varphi'' \\
& \qquad - (2FC_o + \frac{F^2}{EA_c}) z_o (z_o \varphi)'' + k\, \varphi + M_y \upsilon'' = 0 \\
& \textit{where}: \quad C_o = \frac{\sqrt{\ell^2 - 16 f^2}}{\ell} \quad \textit{and thus} \quad F_x = F \cos\varphi = F\, C_o
\end{aligned}
\right\}
$$

(14)

Practically, it is usually: $C_o \cong 1$.

4 NUMERICAL RESULTS

It is obvious, that Eqn. 14 constitute a non linear system which cannot be solved through elementary methods.

We will try to solve the above system through an approaching method for an usual case of a free beam (k=0), with cross-section of double symmetry, prestressed by a rectilinear cable and loaded by a pair of moments M_y=constant, acting at its ends (see Fig.7).

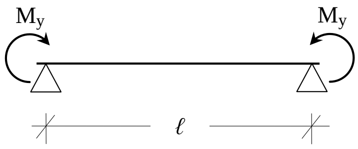


Figure 7. Steel beam loaded by a pair of moments My=constant, acting at its ends

For this purpose we will use a beam of different lengths having a cross-section shown in Fig. 8.

Easily, one can find that the above beam has the following data:

$$I_y = 0.02082\, m^4, \quad I_z = 0.00032\, m^4, \quad I_\omega = 0.0002048\, m^6, \quad A_b = 0.0496\, m^2$$

$$I_d = 9.38 \cdot 10^{-6}\ m^4$$

For the prestressed cables we have:

$$\sigma_c = 12 \cdot 10^6\ kN/m^2,\ z_o = e_z \quad and \quad A_c = 0.001\, m^2$$

Beam and cables have the same modulus of elasticity: $E = 2.1 \cdot 10^8\ kN/m^2$.

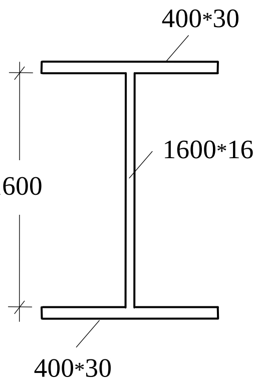


Figure 8. Cross-Section of the beam

Given that $z_M = 0$, $z_o = e_z = constant$, $V_y = 0$, $k = 0$, and that the cable is rectilinear, equations (14) get the following form:

$$\left.\begin{aligned} &E\,I_z \upsilon'''' - (F - \frac{F^2}{EA_c})\upsilon'' + (F + \frac{F^2}{EA_c})\, e_z \varphi'' + M_y \varphi'' = 0 \\ &E\,I_\omega \varphi'''' - (G\,I_d - i_M^2 F)\varphi'' + e_z\,(F + \frac{F^2}{EA_c})\upsilon'' - (2F + \frac{F^2}{EA_c}) e_z^2 \varphi'' + M_y \upsilon'' = 0 \end{aligned}\right\} \tag{15}$$

We consider solutions of the form:

$$\left.\begin{aligned} \upsilon &= V \cdot Sin \frac{\pi\, x}{\ell} \\ \varphi &= \Phi \cdot Sin \frac{\pi\, x}{\ell} \end{aligned}\right\} \tag{16a,b}$$

Introducing (16) into (15), we get:

$$\left.\begin{aligned} &E\,I_z \left(\frac{\pi}{\ell}\right)^4 V + (F - \frac{F^2}{EA_c})\left(\frac{\pi}{\ell}\right)^2 V - (F + \frac{F^2}{EA_c})\, e_z \left(\frac{\pi}{\ell}\right)^2 \Phi - M_y \left(\frac{\pi}{\ell}\right)^2 \Phi = 0 \\ &E\,I_\omega \left(\frac{\pi}{\ell}\right)^4 \Phi + (G\,I_d - i_M^2 F)\left(\frac{\pi}{\ell}\right)^2 \Phi - e_z\,(F + \frac{F^2}{EA_c})\left(\frac{\pi}{\ell}\right)^2 V \\ &\qquad + (2F + \frac{F^2}{EA_c})\, e_z^2 \left(\frac{\pi}{\ell}\right)^2 \Phi - M_y \left(\frac{\pi}{\ell}\right)^2 V = 0 \end{aligned}\right\} \tag{17}$$

Taking into account that:

$$\left.\begin{aligned} &E\,I_z \left(\frac{\pi}{\ell}\right)^2 = P_e \\ &\frac{1}{i_M^2}\left[E\,I_\omega \left(\frac{\pi}{\ell}\right)^2 + G\,I_d\right] = P_T \end{aligned}\right\} \tag{18}$$

are the critical loads of pure buckling and of lateral-torsional buckling respectively, Eqns (17) conclude to the following ones:

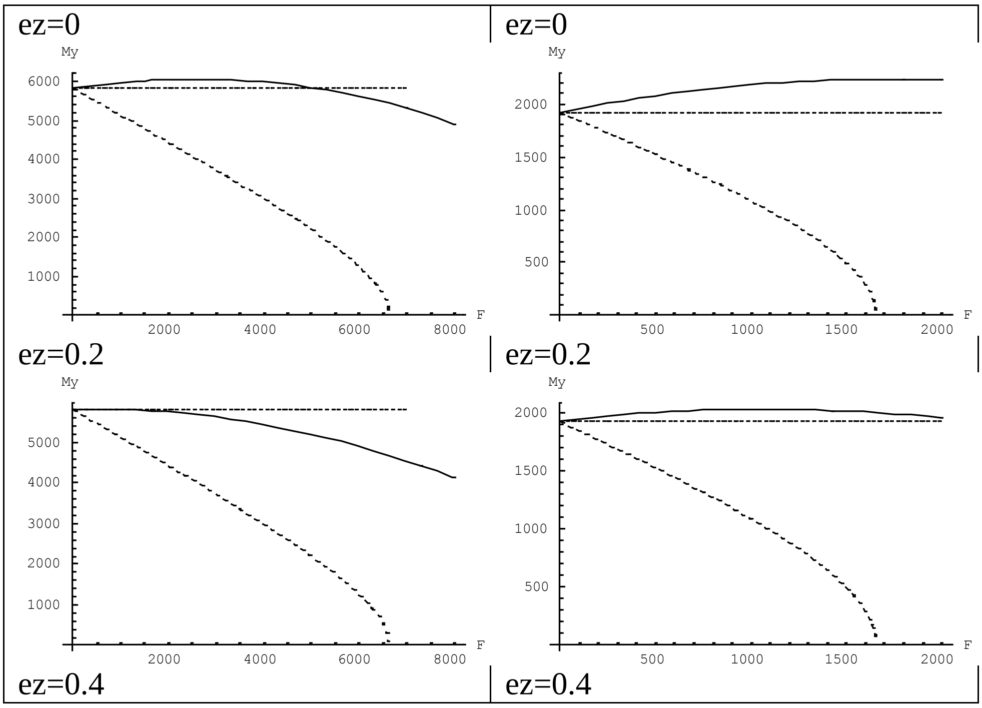
$$\left.\begin{aligned} &A \cdot V + (B - M_y) \cdot \Phi = 0 \\ &(\Gamma - M_y) \cdot V + \Delta \cdot \Phi = 0 \end{aligned}\right\} \tag{19a}$$

Where:

$$\left.\begin{aligned} A &= P_e + F - \frac{F^2}{E\,A_c} \\ B &= -e_z\,(F + \frac{F^2}{E\,A_c}) \\ \Gamma &= B \\ \Delta &= i_M^2\,(P_T - F) + e_z^2\,(2F + \frac{F^2}{E\,A_c}) \end{aligned}\right\} \qquad (19b)$$

In order for the system of Eqn. 19a to have not only trivial solutions, the determinant of the coefficients of the unknown must be equal to zero. This condition gives the moment M_y as follows:

$$M_y = \frac{1}{2}\left((B+\Gamma) + \sqrt{(B-\Gamma)^2 + 4A\Delta}\right) \qquad (20)$$



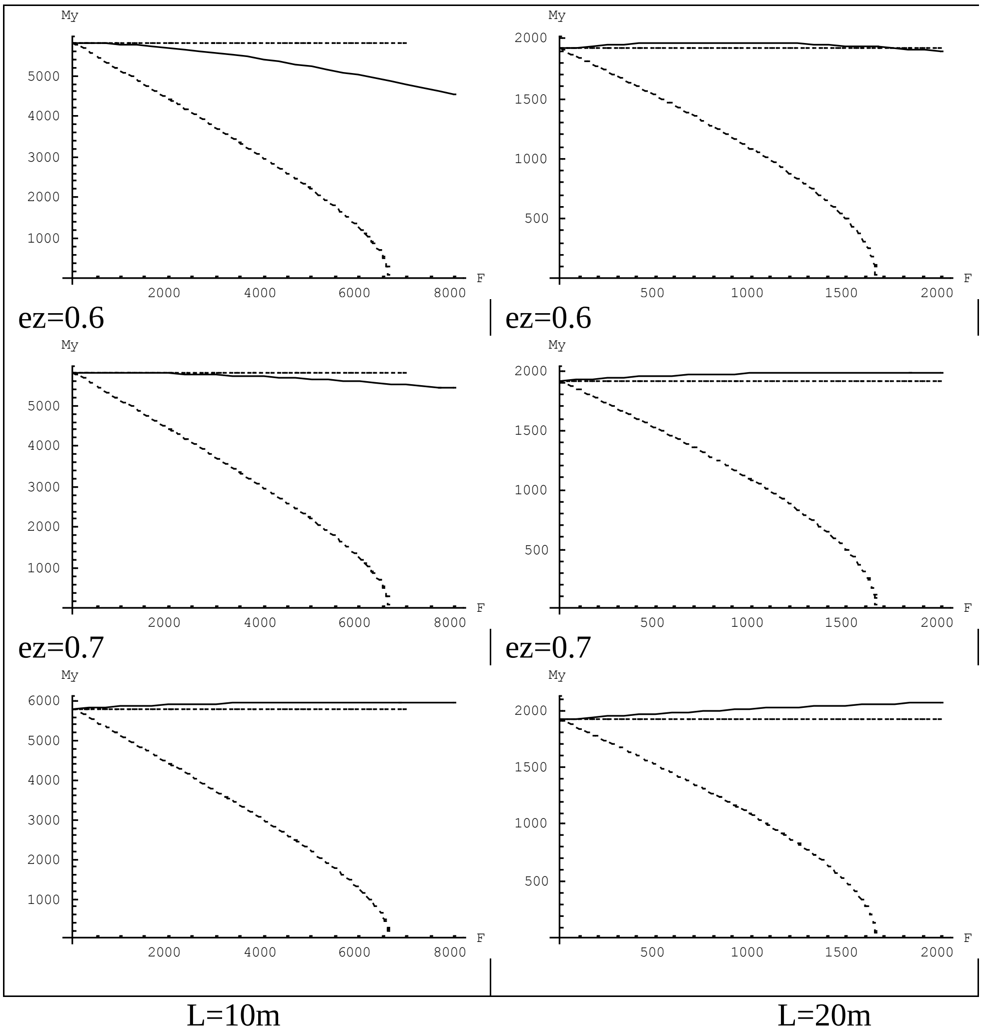


Figure 9. ..…Beam without F, - - - Beam with load f but without prestressing
_____ Beam with prestressing force F

In the above relations taking F=0 we get the critical load when M_y acts alone, while taking z_o =0, we get the critical M_y for simultaneously action of load F (with out prestress):

$$\left.\begin{aligned} M_y &= i_M \sqrt{P_e \cdot P_T} \\ M_y &= i_M \sqrt{(P_e - F)\cdot(P_T - F)} \end{aligned}\right\} \qquad (21)$$

Applying the above data, for beams with length 10 and 20 m k=10, and different e_z, we get the plots of Fig. 9.

5 CONCLUSIONS

From these plots we point out the following:

- Prestressing increases the ability of a beam against the twisting phenomenon.
- This increase depends on the eccentricity of the rectilinear tendon, but, in any way, it is particularly significant, while sometimes surpasses even the ability of the beam in pure buckling due to a pair of M_y acting alone.

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