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THE EFFECT OF STEREOTOMY ON THE SHAPE OF THE THRUST-LINE AND THE MINIMUM THICKNESS OF MASONRY ARCHES

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ABSTRACT: More than a century ago Milankovitch presented for the first time the correct and complete solution for the theoretical minimum thickness, *t*, of a semicircular masonry arch with radius, *R* (t/R=0.1075). This paper uses a variational formulation approach and shows that this solution is not unique and that it depends on the stereotomy exercised.

KEY WORDS: Catenary curve; Limit analysis; Lines of resistance; Stone arches; Variational formulation

1 INTRODUCTION

Robert Hooke [1] was apparently the first to propose a rational rule for sizing masonry arches by describing the analogy in the load-path between a "hanging chain", which forms a catenary in tension under its own weight and a masonry arch which stands under compression. This analogy conceived by Hooke is expressed in the literature "As hangs the flexible line, so but inverted will stand the rigid arch" ([2-4]). The problem of determining the minimum thickness of masonry arches has challenged the engineering community since the early eighteen century (Couplet [5]), was tackled with remarkable ingenuity by Monasterio in the early nineteen century ([6,7]), was addressed rigorously in the early twenty century in the nearly unknown work by Milankovitch [8,9] and remains worth discussing until today (Heyman [10]).

This paper shows that Milankovitch's [8,9] solution for the minimum thickness of a semicircular arch t/R=0.1075 is not unique and that it depends on the stereotomy exercised and the associated coordinate system adopted. The adoption of vertical cuts, first introduced by Lamé and Clapeyron [11] (see also Timoshenko [12]) and an associated cartesian coordinate system yields a slightly higher value for the minimum thickness (t/R=0.1095) than the one computed by Milankovitch. Furthermore, the paper shows that Heyman's [13] widely accepted solution (t/R=0.106) remains unconservative regardless the stereotomy exercised on the arch—even if one assumes vertical joints (Heyman [10]).

2 PHYSICALLY ADMISSIBLE THRUST-LINES OF AN ARCHED MONOLITH WITH ZERO TENSILE STRENGTH

The remarkable directness of Hooke's [1] "hanging chain" as a design tool for sizing stable masonry arches is probably the reason that even in the recent literature it is widely believed that the catenary (the alysoid) is a physically admissible thrust-line of the masonry arch. The thrust-line (or the *line of resistance*, Moseley [14], or the *druckkurve*, Milankovitch [8,9]) is defined as the geometrical locus of the application points of the resultant thrust-force that develops at any cross section of the arch. Given that the calculation of the minimum thickness of the masonry arch derives from the limiting arch that is thick enough so that it can just accommodate a physically admissible thrust-line, the identification of physical admissible thrust-lines is central in this study. *Fig. 1* (left) plots two different minimum thrust-lines within a monolithic elliptical arch with b/a=0.75 and t/a=0.15 (a and b are the semispan and the height of the arch respectively). They have been constructed with a custom-made computer code which repeats the force equilibrium as point O runs from the crown to the springing. These two physically admissible thrust-lines are not distinguishable in the scale of the arch; yet, if one zooms in the neighbourhood isolated with the dashed parallelogram, the two thrust-lines are clearly different as shown in *Fig. 1* (right). Both minimum thrust-lines are equally correct and the fact that they lie within the physical boundaries of the arch ensures that the arch is stable (see also Alexakis and Makris [16]).

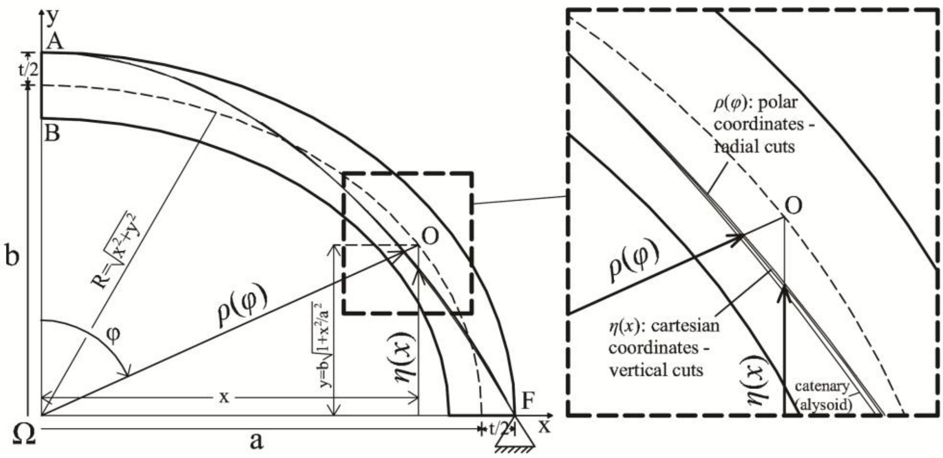

Figure 1. Monolithic elliptical arch with the two different physically admissible minimum thrust-lines (left), b/a=0.75 and t/a=0.15 (a=semispan and b=height). The two physically admissible thrust-lines ($\rho(\varphi)$ obtained with successive radial cuts and $\eta(x)$ obtained with successive vertical cuts) are distinguishable in the enlarged parallelogram (right). The catenary (alysoid) that passes by points A and F is not a physically admissible thrust-line.

In structural engineering the derivation of multiple solutions from equilibrium analysis is not common. The "counterintuitive" result of having two different yet neighboring theoretically correct answers for the thrust-line depending on the coordinate system adopted, derives from the request to express the load path in a two-dimensional structure with finite thickness, *t*, with the thrust-line—that is a concept inherent to a one-dimensional structure (the "hanging chain"). Interestingly, the inverted "hanging chain" (catenary) that passes from the extreme points A and F of the extrados of the arch shown in *Fig. 1* offers a third line that is different from the two minimum thrust-lines—the one computed by taking radial cuts after adopting a polar coordinate system and the other computed with vertical cuts after adopting a cartesian coordinate system.

The idea of analyzing the stability of masonry arches by taking vertical cuts rather than radial cuts goes back to the seminal work of Lamé and Clapeyron [11], who showed that for symmetrical arches of any shape, the calculation of the position of the intrados hinge can be greatly simplified if instead of radial cross-sections, vertical cross-sections are contemplated ([12]).

3 MINIMUM THICKNESS OF A SEMICIRCULAR MONOLITH WITH ZERO TENSILE AND INFINITE COMPRESSION STRENGTH

3.1 Polar coordinate system—Solution with a variational formulation

Milankovitch [8,9] computed the correct value of the minimum thickness of the arched monolith, t/R=0.10748 by deriving the closed-form expression of the minimum thrust-line in association with the information that when the circular monolith assumes its minimum thickness, the minimum thrust-line also touches the intrados of the arch.

When the thickness of the arch is sufficiently reduced and the minimum thrust-line touches the intrados, the arch reaches a limit-equilibrium state by developing a five-hinge symmetric mechanism (Couplet [5]). Accordingly, points A, K, and F shown in *Fig. 2* (left) are imminent hinges of the arch at its limit equilibrium state. When assuming a rupture along the radial direction (that is consistent with the polar coordinate system adopted by Milankovitch [8,9]) moment equilibrium of half the arch and of the top hinged portion of the arch ABK about hinge K gives (Makris and Alexakis [17,18], Alexakis [19])

$$[2\frac{t}{R}(\frac{t^2}{R^2}+12)-3\pi(\frac{t^2}{R^2}-4)]\cos\phi_r = 3\pi(\frac{t}{R}+2)^2+6(\frac{t^2}{R^2}-4)\phi_r\sin\phi_r \quad (1)$$

where φ_r is the rupture angle (see *Fig. 2* left). Adopting as a reference level the horizontal axis *x* (y=0), the potential energy of the semicircular arch is

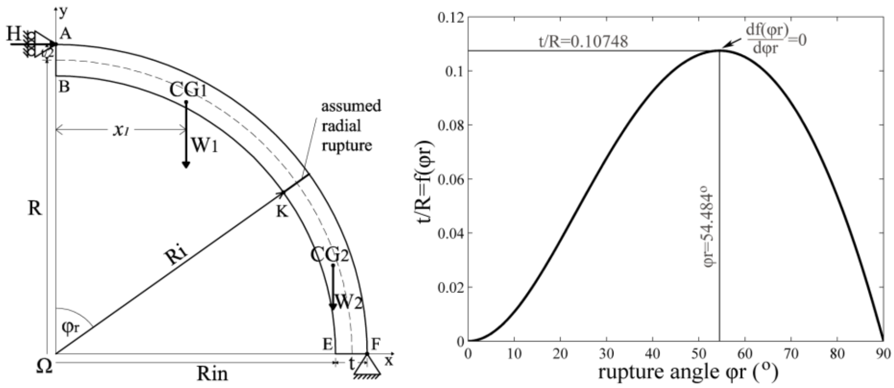

Figure 2. Left: Formation of a hinge mechanism by assuming that the rupture at the intrados hinge, K, happens along the radial direction. Right: Admissible values of t/R for a given rupture angle φ_r as they result from moment equilibrium of the hinged mechanism shown in *Fig. 2* left. The principle of stationary potential energy ($\delta V(\varphi_r)=0$) indicates that the hinged semicircular arch with a radial rupture is in equilibrium when $df(\varphi_r)/d\varphi_r=0$. This happens when $\varphi_r=54.484^o$ and the corresponding minimum thickness is $t/R=0.10748$.

$$V = W \cdot y_o = R^3 \frac{t}{R}(1+\frac{1}{12}\frac{t^2}{R^2}) \tag{2}$$

where $W=(\pi/2)tR$ is the weight and $y_o=R(12+(t/R)^2)/6\pi$ is the vertical coordinate of the center of gravity of half the arch. When a radial rupture has been assumed the two unknowns t/R and φ_r are related via moment equilibrium with a relation $t/R=f(\varphi_r)$, which represents the accepted root of Eq. (1) ([17-19]). Accordingly, the potential energy of the circular arch given by Eq. (2) can be expressed as a function of φ_r

$$V(\phi_r) = R^3 f(\phi_r) \cdot (1+\frac{1}{12} f(\phi_r)^2) \tag{3}$$

The structural system shown in *Fig. 2* (left) is a typical case where the only forces (weights) acting in the system are conservative and where the work of all forces is accounted by the potential energy $V(\varphi_r)$ given by Eq. (3). According to the principle of stationary potential energy, the geometrically admissible hinged mechanism shown in *Fig. 2* (left) is in an equilibrium state if and only if the total potential energy is stationary, i.e.

$$\delta V(\phi_r) = \frac{dV(\phi_r)}{d\phi_r}\delta\phi_r = 0 \tag{4}$$

Substitution of Eq. (3) into Eq. (4) gives

$$\frac{df(\phi_r)}{d\phi_r}[1+\frac{1}{4}f(\phi_r)^2)]=0 \tag{5}$$

The quantity in brackets in Eq. (5) is always positive; therefore, Eq. (5) is satisfied when $df(\varphi_r)/d\varphi_r=0$. The result of Eq. (5) shows that the symmetric hinged arch is in an equilibrium state ($\delta V(\varphi_r)=0$) if and only if

$$\frac{df(\phi_r)}{d\phi_r}=\frac{d(t/R)}{d\phi_r}=0 \tag{6}$$

The solution of Eq. (6) offers the unknown location of the rupture angle $\varphi_r=54.484^o$ which is precisely the value computed by Milankovitch [8,9]. Substitution of the value of the rupture angle $\varphi_r=54.484^o$ (that is for a rupture assumed along the radial direction) into the acceptable root of the moment equilibrium of Eq. (1) one obtains the minimum thickness value $t/R=0.10748$. *Fig. 2* (right) shows that the maximum of equation $df(\varphi_r)/d\varphi_r=0$ happens at $\varphi_r=54.484^o$ and that the maximum value of the thickness is $t/R=0.10748$.

Heyman's [13] "work-balance" concept was implemented by Ochsendorf [15] who developed a trial-and-error procedure by selecting successive values of the rupture angle φ_r together with the corresponding values of t/R until the "work-balance" equation is satisfied. Ochsendorf [15] assumed that the rupture of the arch at the intrados hinge is along the radial direction and his trial-and-error procedure converged to the correct value of the rupture angle $\varphi_r=54.5^o$ and minimum thickness $t/R=0.1075$ initially discovered by Milankovitch [8,9].

3.2 Cartesian coordinate system - Solution with a variational formulation

In this section we consider again that the circular arch has reached its limit equilibrium state by developing a five-hinge symmetric mechanism. Accordingly, points A, K, and F shown in *Fig. 3* (left) are imminent hinges of the arch at its limit equilibrium state. We now assume that the rupture happens along the vertical direction that is consistent with a cartesian coordinate system. For a vertical rupture the weight W_1 of the hinged portion of the semicircular arch ABK and the abscissa x_1 of their center of gravity are

$$W_1=R^2\frac{1}{2}[\frac{x_r}{R}(\sqrt{c_1^2-\frac{x_r^2}{R^2}}-\sqrt{c_2^2-\frac{x_r^2}{R^2}})+c_1^2\arcsin(\frac{x_r}{Rc_1})-c_2^2\arcsin(\frac{x_r}{Rc_2})] \tag{7}$$

$$x_1=R\frac{2\frac{t}{R}(1+\frac{1}{12}\frac{t^2}{R^2})-\frac{2}{3}[(c_1^2-\frac{x_r^2}{R^2})^{3/2}-(c_2^2-\frac{x_r^2}{R^2})^{3/2}]}{\frac{x_r}{R}(\sqrt{c_1^2-\frac{x_r^2}{R^2}}-\sqrt{c_2^2-\frac{x_r^2}{R^2}})+c_1^2\arcsin(\frac{x_r}{Rc_1})-c_2^2\arcsin(\frac{x_r}{Rc_2})} \tag{8}$$

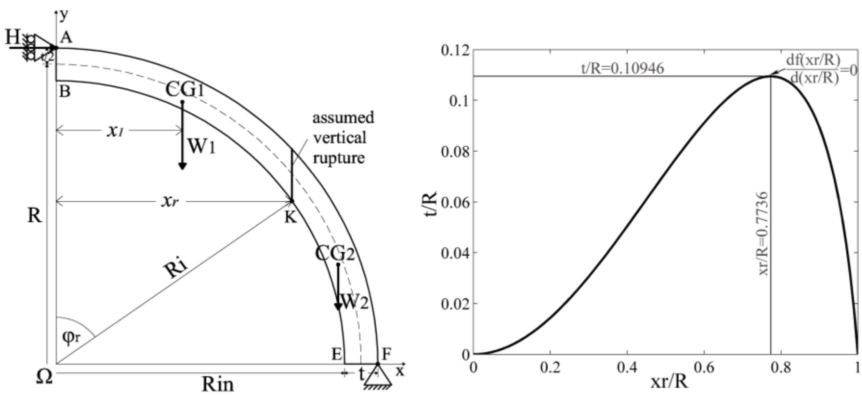

Figure 3. Left: Formation of a hinge mechanism by assuming that the rupture at the intrados hinge K happens along the vertical direction. Right: Admissible values of t/R for a given rupture location x_r/R as they result from moment equilibrium of the hinged mechanism shown in *Fig. 3* left. The principle of stationary potential energy ($\delta V(x_r/R)=0$) indicates that the hinged semicircular arch with a vertical rupture is in equilibrium when $\mathrm{d}f(x_r/R)/\mathrm{d}(x_r/R)=0$. This happens when $x_r=0.7736R$ and the corresponding minimum thickness is $t/R=0.10946$.

In Eqs. (7) and (8), $x_r=Rc_2\sin(\varphi_r)$ is the unknown abscissa of the rupture of the arch; while, c_1 and c_2 are the dimensionless quantities

$$c_1 = 1+\frac{1}{2}\frac{t}{R}\,,\ c_2 = 1-\frac{1}{2}\frac{t}{R} \tag{9}$$

With reference to *Fig. 3* (left), moment equilibrium of the top portion of the arch ABK about hinge K gives

$$H(R+\frac{t}{2}-y_{in}(x_r)) = W_1(x_r-x_1) \tag{10}$$

while moment equilibrium of the entire half arch about hinge F gives the expression of the horizontal thrust-force at the crown H

$$\frac{H}{W} = 1-\frac{1}{\pi}\frac{4+\frac{1}{3}\frac{t^2}{R^2}}{2+\frac{t}{R}} \tag{11}$$

The calculus that follows is identical to the calculus that has been presented in the case of a polar coordinate system (Eqs. (2) to (5)); therefore, the symmetric hinged arch with a vertical rupture at point K is in an equilibrium state ($\delta V(x_r/R)=0$) if and only if

$$\frac{df\left(\frac{x_r}{R}\right)}{d\frac{x_r}{R}} = \frac{d\left(\frac{t}{R}\right)}{d\frac{x_r}{R}} = 0 \tag{12}$$

Numerical processing of the graph appearing in *Fig. 3* (right) shows that the maximum of the graph d$f(x_r/R)$/d(x_r/R) happens at x_r=0.7736R which corresponds to a rupture angle φ_r=BΩK=54.923^o and a minimum thickness value t/R=0.10946. Heyman [13] incorrectly assumed that the line of action of the resultant thrust-force is tangent to the intrados of the arch—therefore tangent to the minimum thrust-line at point K; and this was the origin of the slight error in the calculation of his minimum thickness, t/R=0.106; rather than the correct value t/R=0.1075 ([10,16-20]). Furthermore, Heyman's [13] solution remains unconservative regardless the stereotomy exercised on the arch—even if one assumes vertical joints (Heyman [10]).

The results for the minimum allowable thickness and the rupture location of a semicircular monolith with zero tensile strength subjected to its own weight is summarized in Table 1 together with the list of past publications which derived the correct results with various approaches.

Table 1. Minimum allowable thickness and rupture locations of a semicircular monolith with zero tensile strength

Radial cuts – Polar Coordinate System		Vertical cuts – Cartesian Coordinate System	
Rupture angle φ_r=BΩK=54.484^o	Minimum thickness t/R=0.10748	Rupture angle φ_r=BΩK=54.923^o	Minimum thickness t/R=0.10946
M. Milankovitch [8,9]: Geometric Solution J. Ochsendorf [15]: Trial-and-error solution of the "work balance" equation This work: Principle of Stat. Potential Energy		This work: Principle of Stationary Potential Energy	

4 CONCLUSIONS

This paper revisits the limit equilibrium analysis of a semicircular monolith with zero tensile strength and radius R. When the monolith assumes its minimum thickness t, a symmetric five hinge mechanism is imminent and at this state any physically admissible thrust-line shall pass by the extrados springing points and be tangent to the extrados at the center of the crown of the arch. The paper shows that Milankovitch's [8,9] solution, t/R=0.1075, is not unique and that it depends on the stereotomy exercised and the associated coordinate system. The adoption of a cartesian coordinate system yields a neighboring thrust-line and a different, slightly higher value for the minimum thickness (t/R=0.1095). This result has been obtained in this paper with a variational formulation which emerges as a powerful analysis tool that is liberated from the concept of the thrust-line ([16-20]).

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